Notes on Rational Choice

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Preface

These notes supplement the reading for the course PHPE 400 - Individual and Group Decision Making. Topics include individual decisions (decision theory), strategic decisions (game theory) and group decisions (social choice theory).

> **!** Important
>
> These notes will be continually updated throughout the semester. Please check back regularly.

Part I

Mathematical Preliminaries

This section is a brief introduction to the mathematical concepts and notation that will be used in this course.

Additional reading that covers this material:

- Chapters 1 - 3 from *Mathematical Methods in Linguistics*
- Khan Academy videos about set theory
- Interactive introduction to set theory
- Interactive introduction to functions

Chapter 1

Sets

In this chapter, we introduce the notation and terminology about sets that will be used this semester. There are two ways to write down a set:

1. List all the items in the set. The items should be separated by a comma and the list should be written between curly brackets: '{' and '}'. For example, $\{a, b, c, e\}$ is the set consisting of the first 5 letters of the alphabet, and $\{1, 3, 5\}$ is the set consisting of the first 3 odd numbers.
2. Define a property that *all* items in the set have in common. This is useful when listing all the elements of the set is too cumbersome or impossible. For example, we denote the set of all positive integers by $\{x \mid x \text{ is an integer and } x \geq 0\}$. This is read "the set of all x such that x is an integer and x is greater than or equal to zero".

> **Note**
>
> We sometimes use ellipses '...' when the common property of the elements of a set can be inferred. For instance, the following are equivalent ways to denote the set consisting of the first 10 positive integers:
> - $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
> - $\{1, 2, \ldots, 10\}$
> - $\{x \mid x \text{ is an integer and } 1 \leq x \leq 10\}$

There are two ways that we can talk about what is *contained* in a set.

Definition 1.1 (Element)**.** Suppose that A is a set. We write $x \in A$ when x is a **element** of A and $x \notin A$ when x is not an element of A.

Definition 1.2 (Subset)**.** Suppose that A and B are both sets. We say that:

- A is a **subset** of B, denoted $A \subseteq B$, provided that for all x, if $x \in A$, then $x \in B$;
- A is **not a subset** of B, denoted $A \nsubseteq B$, provided that there is some $x \in A$ such that $x \notin B$; and
- A is a **proper subset** of B, denoted $A \subsetneq B$, provided that $A \subseteq B$ and there is some x such that $x \in B$ and $x \notin A$.

The following examples illustrate the above definitions:

- $0 \in \{0, 1, 3\}$
- $2 \notin \{0, 1, 3\}$
- $\{0, 1, 3\} \subseteq \{0, 1, 2, 3, 4\}$

- $\{0, 1, 3\} \not\subseteq \{0, 2, 4\}$
- $\{0, 1, 3\} \subseteq \{0, 1, 3\}$
- $\{0, 1\} \subsetneq \{0, 1, 3\}$

It is important to remember that the notation "$A \subseteq B$" should only be used when A and B are both sets. For example, if $X = \{1, 2, 3\}$, then it is incorrect to write "$1 \subseteq X$" since 1 is not a set.

> **Warning**
>
> We use the phrase "…is contained in" when talking about both elements and subsets. If $A = \{1, 2, 3\}$, then it is common to say that "A contains 1" or "1 is contained in A". Something that can be confusing for beginners is that it is also common to say that "the set $\{1, 2\}$ is contained in A" since each element in $\{1, 2\}$ is contained in A.

It will be useful to introduce notation for a set containing no elements.

Definition 1.3 (Empty Set)**.** The set that contains no elements is called the **emptyset**, or **null set**. We write $\emptyset$ to denote the emptyset.

A key fact about the empty set is that it is a subset of any set: that is, for all sets X, $\emptyset \subseteq X$. The reason why $\emptyset \subseteq X$ for any set X is that since $\emptyset$ does not contain any elements, there is no element that is contained in $\emptyset$ but not in X.

Note that it is possible for a set to contain an element that is itself a set. That is, a set can contain other sets as members. For instance, the set $X = \{a, \{b, c\}\}$ contains two elements: $a \in X$ and $\{b, c\} \in X$; on the other hand, the set $Y = \{a, b, c\}$ contains three elements: $a \in Y$, $b \in Y$ and $c \in Y$.

Given a set A, we will often be interested in the set of all subsets of A:

Definition 1.4 (Powerset)**.** The **power set** of a set A is the set of all subsets of A. If A is a set, then the power set of A is the set $\wp(A) = \{B \mid B \subseteq A\}$.

Note that $\emptyset \in \wp(A)$ for any set A. To illustrate the power set operation, suppose that $A = \{1, 2, 3\}$. Then the power set of A is:

$$\wp(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Finally, it will be useful to introduce notation to represent the number of elements in a set.

Definition 1.5 (Cardinality)**.** : The cardinality of a finite set A, denoted $|A|$, is the number of elements in A.

> **Note**
>
> The notion of cardinality can be applied to infinite sets as well. However, a discussion of this is beyond the scope of these introductory notes.

https://www.youtube.com/embed/LrLFteztomE

1.1 Notation

The following sets will be discussed this semester.

Notation	Set
$\mathbb{N}$	The set of all non-negative integers $\{0, 1, 2, ...\}$
$\mathbb{Z}$	The set of all integers $\{..., -2, -1, 0, 1, 2, ...\}$
$\mathbb{R}$	The set of all real numbers
$[0, 1]$	The set of all real numbers between (and including) 0 and 1: $[0, 1] = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 1\}$
$(0, 1)$	The set of all real numbers strictly between 0 and 1: $(0, 1) = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$
$\mathbb{Q}$	The set of all rational numbers $\{x \mid x = \frac{a}{b}$, where a, b are integers$\}$

1.2 Exercises

1. Suppose that $X = \{a\}$ and $Y = \{a, c\}$.
   a. True or False: $\emptyset \in X$
   b. True or False: $\emptyset \subseteq X$
   c. True or False: $X \subseteq \emptyset$
   d. True or False: $a \in \emptyset$
   e. True or False: $a \in X$
   f. True or False: $a \subseteq X$.
   g. True or False: $\{a, b\} \subseteq Y$.
   h. True or False: $a \in \{X, Y\}$.
   i. True or False: $\{a\} \subseteq \{X, Y\}$.
   j. True or False: $\{a\} \in \{X, Y\}$.
2. Answer the following:
   a. True or False: $\emptyset \subseteq \emptyset$
   b. True or False: $\mathbb{Z} \subseteq \mathbb{N}$
   c. True or False: $(0, 1) \subseteq \mathbb{Q}$
   d. True or False: $[0, 1] \subseteq (0, 1)$
   e. True or False: $(0, 1) \subseteq [0, 1]$
3. Consider the sets $A = \{\emptyset\}$, $B = \{A\}$ and $C = \{B, \emptyset\}$.
   a. True or False: $\emptyset \in A$.
   b. True or False: $\emptyset \in B$.
   c. True or False: $A \subseteq B$.
   d. True or False: $\emptyset \subseteq B$.
   e. True or False: $\emptyset \in B$.
   f. True or False: $B \in C$.
   g. True or False: $A \in C$.
4. Find the following sets: $\wp(\emptyset)$, $\wp(\{a\})$, and $\wp(\{a, \{a\}\})$.
5. Suppose that A and B are sets.
   a. True or False: For all sets A and B, if $A \subseteq B$, then $|A| \leq |B|$.

   b. True or False: For all sets A and B, if $|A| \leq |B|$, then $A \subseteq B$.
6. Can a set contain an element that is also a subset?
7. Suppose that $|A| = n$. What is $|\wp(A)|$?

Solutions

1. Suppose that $X = \{a\}$ and $Y = \{a, c\}$.
   a. False: $\emptyset \in X$
      The single element of X is a, not $\emptyset$.
   b. True: $\emptyset \subseteq X$
      The empty set is a subset of any set. Since there is no element of $\emptyset$, there is no element of $\emptyset$ that is not an element of X.
   c. False: $X \subseteq \emptyset$
      $a \in X$ but $a \notin \emptyset$.
   d. False: $a \in \emptyset$
      The emptyset $\emptyset$ does not contain any elements.
   e. True: $a \in X$
      $a \in X = \{a\}$
   f. False: $a \subseteq X$.
      a does not contain any elements so cannot be a subset of Y.
   g. False: $\{a, b\} \subseteq Y$.
      $b \in \{a, b\}$. but $b \notin Y = \{a, c\}$.
   h. False: $a \in \{X, Y\}$.
      a is not an element of the set $\{X, Y\} = \{\{a\}, \{a, c\}\}$.
   i. False: $\{a\} \subseteq \{X, Y\}$.
      $a \in \{a\}$, but a is not an element of the set $\{X, Y\} = \{\{a\}, \{a, c\}\}$.
   j. True: $\{a\} \in \{X, Y\}$.
      $\{a\}$ is an element of the set $\{X, Y\} = \{\{a\}, \{a, c\}\}$.
2. Answer the following:
   a. True or False: $\emptyset \subseteq \emptyset$
      True: Since there is no element of $\emptyset$, there is no element of $\emptyset$ that is not an element of $\emptyset$.
   b. True or False: $\mathbb{Z} \subseteq \mathbb{N}$
      False: $-1 \in \mathbb{Z}$, but $-1 \notin \mathbb{N}$.
   c. True or False: $(0, 1) \subseteq \mathbb{Q}$
      False: $\sqrt{2}/2 \in (0, 1)$, but $\sqrt{2}/2 \notin \mathbb{Q}$ (since $\sqrt{2}$ is an irrational number, $\sqrt{2}/2$ cannot be written as a/b where a and b are integers).
   d. True or False: $[0, 1] \subseteq (0, 1)$
      False: $0 \in [0, 1]$, but $0 \notin (0, 1)$.
   e. True or False: $(0, 1) \subseteq [0, 1]$
      True: Every real number strictly between 0 and 1 is an element of $[0, 1]$.

3. Consider the sets $A = \{\emptyset\}$, $B = \{A\}$ and $C = \{B, \emptyset\}$.
   a. True or False: $\emptyset \in A$.
      $\emptyset \in A$ is true: The emptyset is the single element contained in A.
   b. True or False: $\emptyset \in B$.
      $\emptyset \in B$ is false: Note that $B = \{\{\emptyset\}\}$, so B contains the set containing the emptyset, but does not contain $\emptyset$.
   c. True or False: $A \subseteq B$.
      $A \subseteq B$ is false: Since $\emptyset \notin B$, we have $A \nsubseteq B$.
   d. True or False: $\emptyset \subseteq B$.
      $\emptyset \subseteq B$ is true: The emptyset is a subset of every set.
   e. True or False: $\emptyset \in B$.
      $\emptyset \in B$ is false: The single element of B is $\{\emptyset\}$ which is not the empty set.

f. True or False: $B \in C$.
$B \in C$ is true: $C = \{\{\{\emptyset\}\}, \emptyset\}$ and $B = \{\{\emptyset\}\}$.

g. True or False: $A \in C$.
$A \in C$ is false: $C = \{\{\{\emptyset\}\}, \emptyset\}$ and $\{\emptyset\} \notin \{\{\{\emptyset\}\}, \emptyset\}$.

4. Find the following sets: $\wp(\emptyset)$, $\wp(\{a\})$, and $\wp(\{a, \{a\}\})$.
   - $\wp(\emptyset) = \{\emptyset\}$
   - $\wp(\{a\}) = \{\emptyset, \{a\}\}$
   - $\wp(\{a, \{a\}\}) = \{\emptyset, \{a\}, \{\{a\}\}, \{a, \{a\}\}$
5. Suppose that A and B are sets.

   a. True or False: For all sets A and B, if $A \subseteq B$, then $|A| \leq |B|$.
   True: Suppose that $A \subseteq B$. Then every element of A is an element of B. Clearly this means that B has at least as many elements as A, i.e., $|A| \leq |B|$.

   b. True or False: For all sets A and B, if $|A| \leq |B|$, then $A \subseteq B$.
   False: Let $A = \{1\}$ and $B = \{2, 3\}$. Then $|A| = 1 \leq |B| = 2$, but $A \not\subseteq B$.
6. Can a set contain an element that is also a subset?
   Let $Z = \{a, \{a\}\}$. Then, $\{a\} \in Z$ and $\{a\} \subseteq Z$ (since $a \in Z$). So, $\{a\}$ is both an element of Z and a subset of Z.
7. Suppose that $|A| = n$. What is $|\wp(A)|$?
   If $|A| = n$, then $|\wp(A)| = 2^n$. For example, if $|A| = 3$, then the powerset of A has $2^3 = 8$ elements. To illustrate this, suppose that $A = \{a, b, c\}$. Then $|A| = 3$ and

$$|\wp(A)| = |\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}| = 8$$

Chapter 2

Relations

The order of the elements in a set does not matter. That is, $\{a, b\}$ is the same set as $\{b, a\}$ (they both denote the set consisting of two elements a and b). Similarly, listing the same element multiple times does not change a set. So, for instance, $\{a, a, b\}$ is the same set as $\{a, b\}$.

We use '(' and ')' when the order of elements is important. For instance, (a, b) is called an **ordered pair**, or **tuple** of length 2. The first component is a and the second component is b. Since the order in which the elements appear matters, we have that $(a, b) \neq (b, a)$. While there is only one set containing the two elements a and b, there are 4 different ordered pairs that can be constructed using the elements a and b:

$$(a, a) \qquad (a, b) \qquad (b, a) \qquad (b, b).$$

https://www.youtube.com/embed/O-C7G08AQP4?si=PyfbplNr5ez0Fk6c

Definition 2.1 (Product)**.** Suppose that A and B are non-empty sets. The **product** of A and B, denoted $A \times B$, is the set of ordered pairs where the first component comes from A and the second component comes from B. That is,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Suppose that $X = \{a, b, c\}$ and $Y = \{1, 2\}$. Then we have the following:

1. $X \times Y = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.
2. $Y \times X = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.
3. $X \times X = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$.

i Tuples of length greater than 2

Suppose that $X = \{a, b\}$ and $Y = \{d, e\}$, we have that $(X \times X) \times Y$ is

$$\{((a, a), d), ((a, b), d), ((b, a), d), ((b, b), d), ((a, a), e), ((a, b), e), ((b, a), e), ((b, b), e)\}.$$

So, elements of $(X \times X) \times Y$ are tuples where the first component is a tuple of length 2 (where each component is from X) and the second component is an element of Y. Often we will drop the parentheses, writing $X \times X \times Y$, and view the elements of this set as tuples of length 3:

$$\{(a, a, d), (a, b, d), (b, a, d), (b, b, d), (a, a, e), (a, b, e), (b, a, e), (b, b, e)\}.$$

The parentheses can be recovered by associating them to the left.

Definition 2.2 (Relation)**.** A **relation** on a set X is a subset of $X \times X$ (the set of pairs of elements from X). That is, if $R \subseteq X \times X$, then R is a relation on X.

Relations are an important mathematical tool used throughout Philosophy, Political Science, and Economics. You have already studied binary relations during your mathematical eduction: $=, \leq, \geq, <$, and $>$ are all relations on numbers (e.g., the natural numbers $\mathbb{N}$, real numbers $\mathbb{R}$, rational numbers $\mathbb{Q}$, etc.) and $\subseteq$ is a relation on the power set of a set. For example, the binary relation $\geq \subseteq \mathbb{N} \times \mathbb{N}$ is the set

$$\{(a, b) \mid a, b \in \mathbb{N} \text{ and } a \text{ is greater than or equal to } b\}.$$

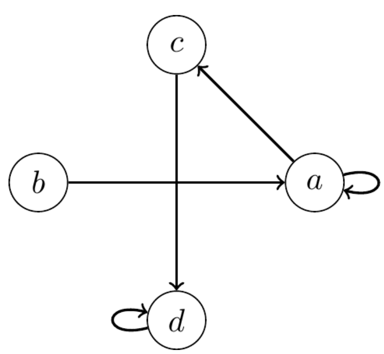
2.1 Notation

Given a set $R \subseteq X \times X$ of ordered pairs, $(a, b) \in R$ represents that *a is related to b according to R*, and if $(a, b) \notin R$, then a is not related to b according to R. To simplify notation, we write $a\ R\ b$ when $(a, b) \in R$. The following is a summary of the notation you will use this semester:

Mathematical Notation	Meaning
$(a, b) \in R$	a is related to b according to R.
$a\ R\ b$	a is related to b according to R.
$(a, b) \notin R$	a is not related to b according to R.
not-$a\ R\ b$	a is not related to b according to R.
$a\ \not R\ b$	a is not related to b according to R.

For instance, we can express that 4 is greater-than-or-equal-to 1 by writing $4 \geq 1$ or $(4, 1) \in \geq$, and that 2 is not greater-than-or-equal-to 3 by writing $2 \not\geq 3$ or $(2, 3) \notin \geq$.

Often it is useful to visualize a relation by drawing arrows between items that are related. To visualize $R \subseteq X \times X$, write down all the elements of X and then for each $(x, y) \in R$ draw an arrow from element x to element y. For example, suppose that $X = \{a, b, c, d\}$ and $R = \{(a, a), (b, a), (c, d), (a, c), (d, d)\}$. Then R is visualized as follows:



For instance, consider the relation $\geq \subseteq \mathbb{N} \times \mathbb{N}$ of "greater-than-or-equal-to". Then, we have that $(4, 1) \in \geq$ since 4 is greater-than-or-equal-to 1 and $(2, 3) \notin \geq$ since 2 is not greater-than-or-equal-to 3.

We will often use the following shorthand to denote elements in the relation: If $x_1, \dots, x_n \in X$, then

$$x_1 \; R \; x_2 \; R \cdots x_{n-1} \; R \; x_n$$

means that for all $i = 1, \dots, n-1$, $(x_i, x_{i+1}) \in R$ or $(x_i, x_j) \in R$ for all $j < i$ if R is assumed to be transitive (or $j \leq i$ if R is assumed to also be reflexive). For example, if R is transitive and reflexive, then $a \; R \; b \; R \; c$ means that $\{(a, a), (a, b), (b, b), (a, c), (b, c), (c, c)\} \subseteq R$.

2.2 Lecture

https://www.youtube.com/embed/KacOh3TFruA

2.3 Properties of Relations

Typically, we are interested in relations satisfying special properties. Suppose that $R \subseteq X \times X$. The following properties of R will be discussed this semester:

- R is **reflexive** provided that for all $a \in X$, $a \; R \; a$.
- R is **connected** provided that for all $a, b \in X$, $a \; R \; b$ or $b \; R \; a$ (or both). A connected relation is also called **complete**.
- R is **symmetric** provided that for all $a, b \in X$, if $a \; R \; b$ then $b \; R \; a$.
- R is **asymmetric** provided that for all $a, b \in X$, if $a \; R \; b$ then it is not the case that $b \; R \; a$.
- R is **transitive** provided that for all $a, b, c \in X$, if $a \; R \; b$ and $b \; R \; c$ then $a \; R \; c$.

> **Note**
>
> As stated, connectedness implies reflexivity (let $a = b$ in the definition of connected). Sometimes, connectedness is defined as follows: for all *distinct* $a, b \in X$, $a \; R \; b$ or $b \; R \; a$. In what follows, we will use the above stronger definition of connected where all connected relations are reflexive.

The following tutorial gives you a chance to practice identifying properties of relations: https://epacuit-relation-properties-relation-properties-q9wf83.streamlitapp.com

2.4 Cycles

Suppose that R is a relation on X. A **path** in R is a sequence of elements $x_1, x_2, \dots, x_n \in X$ such that each element of the sequence is R-related to the next element. That is,

$$x_1 \; R \; x_2 \; R \; x_3 \; R \cdots x_{n-1} \; x_n$$

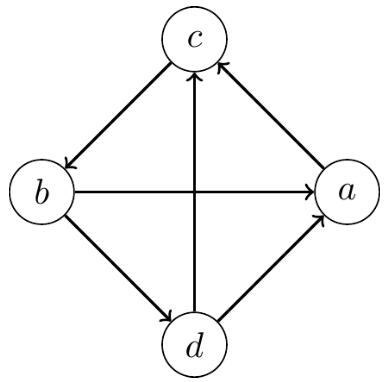
When the last element of a path is also related to the first elements, we call the path a **cycle**.

Definition 2.3 (Cycle)**.** Suppose that $R \subseteq X \times X$. A (simple) cycle in R is a sequence of distinct elements $(x_1, x_2, \dots, x_n)$ such that $x_i \in X$ for all $i = 1, \dots, n$ and for all $i = 1, \dots, n-1$, $x_i \; R \; x_{i+1}$, and $x_n \; R \; x_1$. A relation R is said to be **acyclic** if there are no cycles.

For example, suppose that $X = \{a, b, c, d\}$ and $R = \{(a, c), (b, a), (b, d), (c, b), (d, c), (d, a)\}$. This relation can be pictured as follows:

There are 3 cycles in R:

1. (a, c, b)



2. (c, b, d)
3. (c, b, d, a).

This examples demonstrates that:

- A relation $R \subseteq X \times X$ may have multiple cycles; and
- A cycle in a $R \subseteq X \times X$ need not involve all elements of X.

2.5 Exercises

1. Suppose that $A = \{b, c\}$ and $B = \{2, 3\}$. Find all the following sets.
   a. $A \times B$
   b. $B \times A$
   c. $A \times A$
   d. $B \times B$
   e. $(A \times A) \times B$
2. Suppose that $X = \{a, b, c\}$ and that $R = \{(a, a), (b, c), (a, c)\}$. What properties does R satisfy?
   a. Reflexive
   b. Symmetric
   c. Asymmetric
   d. Connected
   e. Transitive
3. Suppose that $X = \{a, b, c\}$ and that $R = \{(a, b), (b, c), (c, a), (a, a), (b, b), (c, c)\}$. What properties does R satisfy?
   a. Reflexive
   b. Symmetric
   c. Asymmetric
   d. Connected
   e. Transitive
4. Suppose that $X = \{a, b, c\}$ and that $R = \{(a, b), (b, c), (a, c)\}$. What properties does R satisfy?
   a. Reflexive
   b. Symmetric
   c. Asymmetric

d. Connected
e. Transitive

5. Suppose that $X = \{a, b, c\}$ and that $R = \{(a, b)\}$. What properties does R satisfy?

   a. Reflexive
   b. Symmetric
   c. Asymmetric
   d. Connected
   e. Transitive

6. What properties does the relation $\geq$ on numbers satisfy?

7. What properties does the relation $>$ on numbers satisfy?

8. Give an example of a relation on people that satisfies:

   1. transitivity and symmetry.

   2. symmetry but not transitivity.
   3. transitivity but not symmetry.

9. List all the cycles in the relation $R = \{(a, b), (a, c), (a, d), (b, c), (c, d), (d, b)\}$.

10. Can you find an example of a relation that is transitive, symmetric but not reflexive?

Solutions

1. Suppose that $A = \{b, c\}$ and $B = \{2, 3\}$. Find all the following sets.
   a. $A \times B = \{(b, 2), (b, 3), (c, 2), (c, 3)\}$
   b. $B \times A = \{(2, b), (2, c), (3, b), (3, c)\}$
   c. $A \times A = \{(b, b), (b, c), (c, b), (c, c)\}$
   d. $B \times B = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$
   e. $(A \times A) \times B = \{(b, b, 2), (b, c, 2), (c, b, 2), (c, c, 2), (b, b, 3), (b, c, 3), (c, b, 3), (c, c, 3)\}$
2. Suppose that $X = \{a, b, c\}$ and that $R = \{(a, a), (b, c), (a, c)\}$. What properties does R satisfy?
   a. R is not reflexive since $(b, b) \notin R$.
   b. R is not symmetric since $(b, c) \in R$, but $(c, b) \notin R$.
   c. R is not asymmetric since $(a, a) \in R$.
   d. R is not connected since $(b, b) \notin R$.
   e. R is transitive.
3. Suppose that $X = \{a, b, c\}$ and that $R = \{(a, b), (b, c), (c, a), (a, a), (b, b), (c, c)\}$. What properties does R satisfy?
   a. R is reflexive.
   b. R is not symmetric since $(b, c) \in R$, but $(c, b) \notin R$.
   c. R is not asymmetric since $(a, a) \in R$.
   d. R is connected.
   e. R is not transitive since $(a, b) \in R$, $(b, c) \in R$, but $(a, c) \notin R$.
4. Suppose that $X = \{a, b, c\}$ and that $R = \{(a, b), (b, c), (a, c)\}$. What properties does R satisfy?
   a. R is not reflexive since $(a, a) \notin R$.
   b. R is not symmetric since $(b, c) \in R$, but $(c, b) \notin R$.
   c. R is asymmetric.
   d. R is not connected since $(a, a) \notin R$.
   e. R is transitive.
5. Suppose that $X = \{a, b, c\}$ and that $R = \{(a, b)\}$. What properties does R satisfy?

a. R is not reflexive since $(a, a) \notin R$.
b. R is not symmetric since $(a, b) \in R$, but $(b, a) \notin R$.
c. R is asymmetric.
d. R is not connected since $(a, c) \notin R$ and $(c, a) \notin R$.
e. R is transitive.

6. What properties does the relation $\geq$ on numbers satisfy?
   $\geq$ is transitive, reflexive, and connected. $\geq$ is not symmetric since $2 \geq 1$ but $1 \not\geq 2$, and it is not asymmetric since $1 \geq 1$.
7. What properties does the relation $>$ on numbers satisfy?
   $>$ is transitive and asymmetric. $>$ is not reflexive since $1 \not> 1$, it is not symmetric since $2 > 1$ but $1 \not> 2$, and it is not connected since $1 \not> 1$.
8. Give an example of a relation on people that satisfies:
   1. transitivity and symmetry: Consider the relation R in which a person a is related to a person b when a and b have the same last name. (If a hs the same last name as b, then b has the same last name as a, and if a has the same last name as b and b has the same last name as c, then a and c must have the same last name.)

   2. symmetry but not transitivity: Consider the relation R in which a person a is related to a person b and a and b shake hands. Clearly, if a shakes hands with b, then b also shakes hands with a. Suppose that there is a room three people, Ann, Bob, and Charles. Suppose that Ann and Bob shake hands, and Bob and Charles shake hands, but Ann does not shake hands with Charles. This shows that the "shake-hands" relation need not be transitive.
   3. transitivity but not symmetry: Consider the "taller-than" relation in which a is related to b when a is taller than b. It is not hard to see that "taller-than" is transitive, but not symmetric.
9. List all the cycles in the relation $R = \{(a, b), (a, c), (a, d), (b, c), (c, d), (d, b)\}$.
   There is one cycle in this relation: (b, c, d).
10. Can you find an example of a relation that is transitive, symmetric but not reflexive?
    This is tricky. First of all, note that the empty relation $R = \emptyset$ on a non-empty set X is trivially transitive, symmetric, but is not reflexive.
    However, if R is non-empty, then if R is transitive and symmetric, then it is reflexive. Suppose that $(a, b) \in R$. Then, by symmetric $(b, a) \in R$ and by transitivity, $(a, a) \in R$.

Chapter 3

Functions

A **function** from a set A to a set B is a way of associating elements of A with unique elements of B. Formally, a function is a special type of relation:

Definition 3.1 (Function)**.** A function f from A to B is a binary relation on A and B (i.e., $f \subseteq A \times B$) such that for all $a \in A$, there exists a *unique* $b \in B$ such that $(a, b) \in f$. We write $f : A \to B$ when f is a function, and if $(a, b) \in f$, then write $f(a) = b$.

Suppose that $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Examples of relations that are functions include:

- $f_1 = \{(1, a), (2, a), (3, b)\}$. We write $f_1(1) = a$, $f_1(2) = a$, and $f_1(3) = b$
- $f_2 = \{(1, a), (2, a), (3, a)\}$. We write $f_2(1) = a$, $f_2(2) = a$, and $f_2(3) = a$
- $f_3 = \{(1, a), (3, b)\}$. We write $f_3(1) = a$ and $f_3(3) = b$

An example of a relation that is not a function is $R = \{(1, a), (1, b), (3, b)\}$.

When using functions, we will often use the following terminology and notation.

Definition 3.2 (Domain/Codomain)**.** Suppose $f : A \to B$ is a function. The set A is called the **domain** of f and B is called the **codomain** of f.

Definition 3.3 (Image)**.** Suppose $f : A \to B$ is a function. The image of a set $B \subseteq A$ is the set:

$$f(B) = \{a \mid a = f(b) \text{ for some } b \in B\}$$

Definition 3.4 (Inverse Image)**.** : Suppose that $f : A \to B$ and that $Y \subseteq B$. The inverse image of Y is the set

$$f^{-1}(Y) = \{x \mid x \in A \text{ and } f(x) \in Y\}$$

Definition 3.5 (Range)**.** : Suppose that $f : A \to B$. The range of f function is the image of its domain.

For example, suppose that $A = \{a, b, c, d\}$ and $f : A \to \mathbb{N}$ is the function where $f(a) = 1$, $f(b) = 1$, $f(c) = 2$ and $f(d) = 4$. Then, we have the following:

- The domain of f is A
- The codomain of f is $\mathbb{N}$
- The range of f is $\{1, 2\}$
- The image of $\{a, b\}$ is $f(\{a, b\}) = \{1\}$

- The inverse image of $\{2,4\}$ is $f^{-1}(\{2,4\}) = \{c,d\}$

Note

Example 3.1. In rational choice theory, we will often consider functions with domain and/or codomains that are powersets of a set. For example, suppose that $X = \{a,b,c\}$. A function from non-empty subsets of X to non-empty subsets of X is denoted $f : (\wp(X)\ \ \emptyset) \to (\wp(X)\ \ \emptyset)$. An example of such a function is:

$$\begin{aligned} f(\{a\}) &= \{b\} \\ f(\{b\}) &= \{b\} \\ f(\{c\}) &= \{c\} \\ f(\{a,b\}) &= \{a\} \\ f(\{a,c\}) &= \{b\} \\ f(\{b,c\}) &= \{b\} \\ f(\{a,b,c\}) &= \{b,c\} \end{aligned}$$

In many situation, we will need to apply one function to the output of another function. More formally, when the codomain of a function is the same as the domain of another function, then the functions can be composed to form a new function:

Definition 3.6 (Composition of functions)**.** Suppose that $f : A \to B$ and $g : B \to C$. The composition of f with g is the function $(g \circ f) : A \to C$ defined as follows: for all $a \in A$, $g \circ f(a) = g(f(a))$

For example if suppose that $A = \{a,b,c,d\}$ and $f : A \to \mathbb{N}$ is the function where $f(a) = 1$, $f(b) = 1$, $f(c) = 2$ and $f(d) = 4$, and g is the function $g : \mathbb{N} \to \mathbb{N}$ where for all $n \in \mathbb{N}$, $g(n) = 2n+1$. Then $(g \circ f) : A \to \mathbb{N}$ is the function where:

- $(g \circ f)(a) = g(f(a)) = g(1) = 3$,
- $(g \circ f)(b) = g(f(b)) = g(1) = 3$,
- $(g \circ f)(c) = g(f(c)) = g(2) = 5$,
- $(g \circ f)(d) = g(f(d)) = g(4) = 7$,

3.1 Examples using the function notation

In this course, we will discuss a number of different types of functions:

- $u : X \to \mathbb{R}$

  u is a function mapping elements of X to real numbers.

- $p : X \to [0,1]$

  p is a function mapping elements of X to real numbers between 0 and 1. For instance, suppose that $X = \{a,b,c\}$ and $p : X \to [0,1]$ is the function with $p(a) = 0$, $p(b) = 0.5$, $p(c) = 0.5$.

- $p : \wp(X) \to [0,1]$

  p is a function mapping subsets of X to real numbers between 0 and 1. For instance, suppose that $X = \{a,b,c\}$ and $p : \wp(X) \to [0,1]$ is the function with $p(\emptyset) = 0$, $p(\{a\}) = 0$, $p(\{b\}) = 0.5$,

$p(\{c\}) = 0.5$, $p(\{a, b\}) = 0.5$, $p(\{a, c\}) = 0.5$, $p(\{b, c\}) = 1.0$, and $p(\{a, b, c\}) = 1.0$.

3.2 Exercises

1. Suppose that $X = \{a, b, c\}$ and $u : X \to \mathbb{R}$ with $u(a) = 0.5$, $u(b) = 1.0$, $u(c) = 2.0$.

   a. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 5$. What is $f \circ u$?
   b. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. What is $f \circ u$?
   c. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -10x$. What is $f \circ u$?

2. Consider the function f defined in Example 3.1. Does this function satisfy the following constraint: for all $Y \in \wp(X) - \emptyset$, $f(Y) \subseteq Y$? If so, explain why. If not, explain why it fails the constraint and find a function that satisfies the constraint.

Solutions

1. Suppose that $X = \{a, b, c\}$ and $u : X \to \mathbb{R}$ with $u(a) = 0.5$, $u(b) = 1.0$, $u(c) = 2.0$.
   a. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 5$. What is $f \circ u$?
      - $f \circ u(a) = f(u(a)) = f(0.5) = 2 * 0.5 + 5 = 6.0$
      - $f \circ u(b) = f(u(b)) = f(1.0) = 2 * 1.0 + 5 = 7.0$
      - $f \circ u(c) = f(u(c)) = f(2.0) = 2 * 2.0 + 5 = 9.0$
   b. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. What is $f \circ u$?
      - $f \circ u(a) = f(u(a)) = f(0.5) = 0.5^2 = 0.25$
      - $f \circ u(b) = f(u(b)) = f(1.0) = 1.0^2 = 1.0$
      - $f \circ u(c) = f(u(c)) = f(2.0) = 2.0^2 = 4.0$
   c. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -10x$. What is $f \circ u$?
2. Consider the function f defined in Example 3.1. Does this function satisfy the following constraint: for all $Y \in \wp(X) - \emptyset$, $f(Y) \subseteq Y$? If so, explain why. If not, explain why it fails the constraint and find a function that satisfies the constraint.
   The above function does not satisfy this constraint since, for instance, $f(\{a\}) = \{b\} \not\subseteq \{a\}$ (we also have that $f(\{a, c\}) = \{b\} \not\subseteq \{a, c\}$). An example of a function that satisfies the above constraint is:

$$
\begin{aligned}
f(\{a\}) &= \{a\} \\
f(\{b\}) &= \{b\} \\
f(\{c\}) &= \{c\} \\
f(\{a, b\}) &= \{a\} \\
f(\{a, c\}) &= \{c\} \\
f(\{b, c\}) &= \{b\} \\
f(\{a, b, c\}) &= \{b, c\}
\end{aligned}
$$

Chapter 4

Lotteries

Suppose that X is a finite set. Elements of X are called *outcomes* or *prizes*. A lottery on X is a probability function on X.

Definition 4.1 (Lottery)**.** Suppose that X is a finite set. A lottery, or probability, on X is a function $p : X \to [0,1]$ such that $\sum_{x \in X} p(x) = 1$.

> **Note**
>
> There are a number of mathematical details about probability that we are glossing over here. Our discussion in this course is greatly simplified since we assume that the set of objects X is finite.

To simplify notation, we represent a lottery $p : X \to [0,1]$ on a set $X = \{x_1, \dots, x_n\}$ as a list associating each outcome with its probability: $[x_1 : p(x_1), x_2 : p(x_2), \dots, x_n : p(x_n)]$.

For instance, if $X = \{a, b, c\}$, then the following are examples of three lotteries on X:

1. $[a : 0, b : 1, c : 0]$: There is a 100% chance of getting b.
2. $[a : 0.25, b : .35, c : 0.4]$: There is a 25% chance of getting a, 35% chance of getting b, and a 40% chance of getting c.
3. $[a : 0.25, b : .75, c : 0]$: There is a 25% chance of getting a and a 75% chance of getting b.

We will make use of the following notation about lotteries in these notes:

- Lotteries in which one outcome is assigned probability 1 are called *sure-things*. We associate each element of $x \in X$ with the sure-thing lottery $[x : 1]$.
- We may not include outcomes that are assigned probability 0 by a lottery. For instance, suppose that $X = \{a, b, c, d\}$. If we say that $[b : 0.5, c : 0.5]$ is a lottery on X, then this is the lottery $[a : 0, b : 0.5, c : 0.5, d : 0]$.

- If lotteries contain the same prize with different probabilities, we can simplify by summing the probabilities for that prize. For instance, the lottery $[a : 0.2, b : 0.1, a : 0.3, c : 0.1, b : 0.3]$ is the same lottery as $[a : (0.3 + 0.2), b : (0.1 + 0.3), c : 0.1] = [a : 0.5, b : 0.4, c : 0.1]$.

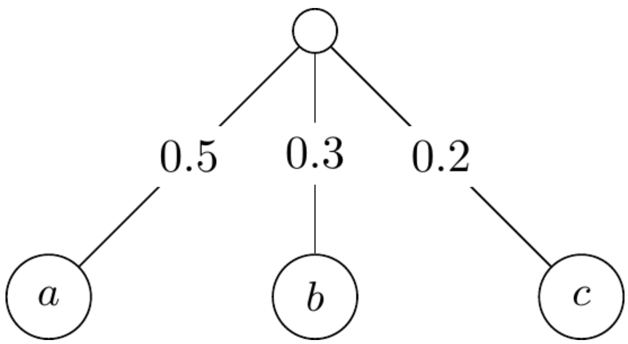
4.1 Compound Lotteries

Suppose that $L_1 = [a : 0.5, b : 0.5]$ and $L_2 = [b : 0.25, c : 0.75]$ are two lotteries on $X = \{a, b, c\}$. Now, consider the lottery in which a fair coin is flipped and if it lands heads, then the lottery L_1 is played, otherwise the lottery L_2 is played. This **compound lottery** can be represented as $[L_1 : 0.5, L_2 : 0.5]$. In fact, any set of lotteries can be combined to form a compound lottery.

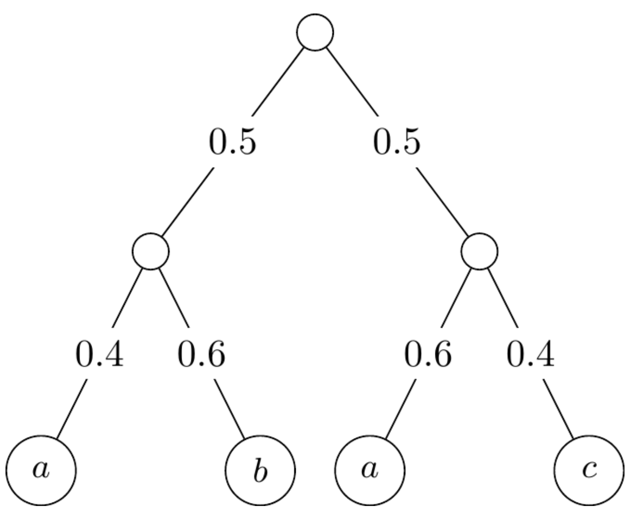
Definition 4.2 (Compound Lottery)**.** Suppose that $L_1, \dots, L_n$ are lotteries on a set X. Then, $[L_1 : p_1, \dots, L_n : p_n]$ is **compound lottery**, where $\sum_{i=1}^{n} p_i = 1$.

We can display compound lotteries as a tree in which $[L_1 : p_1, \dots, L_n : p_n]$ is the tree in which there is an edge from the root node to the tree representing L_i labeled by p_i.

- The lottery $[a : 0.5, b : 0.3, c : 0.2]$ can be pictured as follows:



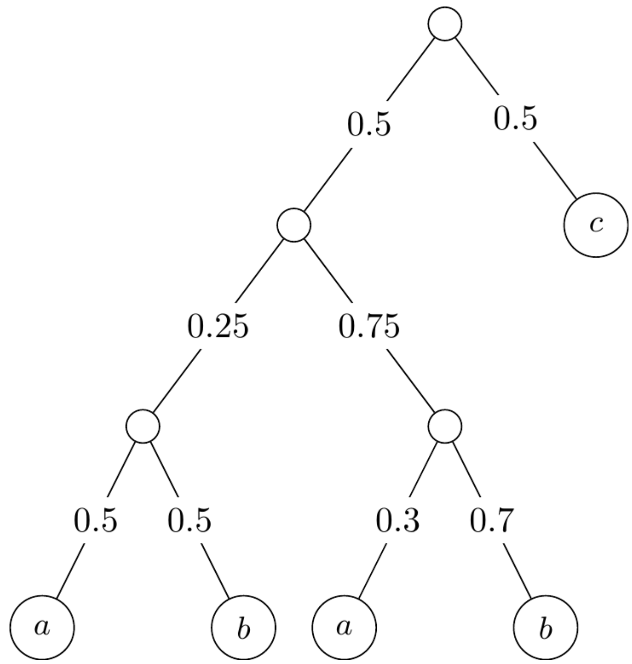
- The lottery $[[a : 0.4, b : 0.6] : 0.5, [a : 0.6, c : 0.4] : 0.5]$ can be pictured as follows:



- The lottery $[[[a : 0.5, b : 0.5] : 0.25, [a : 0.3, b : 0.7] : 0.75] : 0.5, c : 0.5]$ can be pictured as follows:

4.2 Exercises

1. Consider the lottery in which a fair coin is flipped. If it lands heads, then you win \$100 and if it lands tails, you lose \$100. Write this lottery down using the notation described above.



2. Consider the lottery in which a biased coin is flipped. If it lands heads, then you win \$100 and if it lands tails, you lose \$100. Suppose that bias of the coin is that the chance for heads is 3-times the chance for tails. Write this lottery down using the notation described above.

3. Consider the lottery in which a fair coin is flipped. If it lands heads, then the you lose \$5 and if it lands tails, then you roll a fair die (with 6-sides) and you win the amount in dollars shown on the die. Write this lottery down using the notation described above.

4. Consider $[\$10 : 0.5, [\$10 : 0.3, \$5 : 0.7] : 0.5]$. What is the probability that you will \$10? What is the probability that you win \$5?

5. Draw the tree that depicts the following lotteries:

   a. $[a : 0.2, b : 0.4, a : 0.1, c : 0.3]$

   b. $[[a : 0.5, b : 0.5] : 0.2, [a : 1] : 0.8]$
   c. $[[[a : 0.5, b : 0.5] : 0.5, [a : 0.5, b : 0.5] : 0.5] : 0.5, b : 0.5]$

Solutions

1. Consider the lottery in which a fair coin is flipped. If it lands heads, then you win \$100 and if it lands tails, you lose \$100. Write this lottery down using the notation described above.

$$[\$100 : 0.5, -\$100 : 0.5]$$

2. Consider the lottery in which a biased coin is flipped. If it lands heads, then you win \$100 and if it lands tails, you lose \$100. Suppose that bias of the coin is that the chance for heads is 3-times the chance for tails. Write this lottery down using the notation described above.

$$[\$100 : 0.75, -\$100 : 0.25]$$

3. Consider the lottery in which a fair coin is flipped. If it lands heads, then the you lose $5 and if it lands tails, then you roll a fair die (with 6-sides) and you win the amount in dollars shown on the die. Write this lottery down using the notation described above.

$$[-\$5 : \frac{1}{2}, [1 : \frac{1}{6}, 2 : \frac{1}{6}, 3 : \frac{1}{6}, 4 : \frac{1}{6}, 5 : \frac{1}{6}, 6 : \frac{1}{6}] : \frac{1}{2}]$$

4. Consider $[\$10 : 0.5, [\$10 : 0.3, \$5 : 0.7] : 0.5]$. What is the probability that you will $10? What is the probability that you win $5?
   - What is the probability that you will $10? $0.5 + 0.5 * 0.3 = 0.65$
   - What is the probability that you win $5? $0.5 * 0.7 = 0.35$
5. Draw the tree that depicts the following lotteries:

a. $[a : 0.2, b : 0.4, a : 0.1, c : 0.3]$

b. $[[a : 0.5, b : 0.5] : 0.2, [a : 1] : 0.8]$

c. $[[[a : 0.5, b : 0.5] : 0.5, [a : 0.5, b : 0.5] : 0.5] : 0.5, b : 0.5]$

Part II

Preferences

A basic component of any rational choice model is a representation of the decision maker's *preferences*. We start by fixing a non-empty set X representing the feasible alternatives. Elements of X may be the items available to choose, descriptions of the likelihood of obtaining different outcomes, candidates in an election, etc. This chapter describes how to represent a decision maker's preferences about the items in a set X and what it means to assume that a decision maker's preferences are *rational*.

The main readings for this section are:

- Chapter 1: Preferences from Hausman (2012)
- Chapter 2: Preference Axioms and Their Implications from Hausman (2012)
- Chapter 2 from Gaus and Thrasher (2021)

Additional readings about preferences include:

- Section 1 and Section 2 from Hansson and Grüne-Yanoff (2022)

Chapter 5

Preference Relations

Preferences in rational choice theory are understood comparatively. So, if a decision maker "prefers red wine", then this means that the decision maker prefers red wine to the other available alternatives (e.g., red wine more than white wine).

5.1 Strict Preference, Indifference, and Incomparability

Let X be a set of alternatives. A decision maker's *preference* over X is represented by the following relations on X:

1. $P \subseteq X \times X$: for $x, y \in X$, $x\ P\ y$ means that the decision maker **strictly prefers** x to y.
2. $I \subseteq X \times X$: for $x, y \in X$, $x\ I\ y$ means that the decision maker is **indifferent** between x and y.
3. $N \subseteq X \times X$: for $x, y \in X$, $x\ N\ y$ means that the decision maker **cannot compare** x and y.

The first assumption is that the the relations P, I, and N represent the subjective preferences of a single decision maker:

Assumption 1 For all $x, y \in X$, exactly one of $x\ P\ y$, $y\ P\ x$, $x\ I\ y$, or $x\ N\ y$ is true.

Thus, there are four distinct ways a decision maker can compare alternatives x and y:

1. $x\ P\ y$: the decision maker *strictly prefers* x to y.
2. $y\ P\ x$: the decision maker *strictly prefers* y to x.
3. $x\ I\ y$: the decision maker is *indifferent* between x and y.
4. $x\ N\ y$: the decision maker *cannot compare* x and y.

There are additional constraints that we will impose on the relations P, I, and N. The intended interpretation of a strict preference is that if the decision maker strictly prefers x to y (i.e., $x\ P\ y$), then the decision maker would pay some non-zero amount money to trade y for x. Under this interpretation, it is clear that a minimal constraint on P is that it is asymmetric:

Assumption 2 Suppose that $P \subseteq X \times X$ represents the decision maker's strict preferences. We assume that P is asymmetric: for all $x, y \in X$, if $x\ P\ y$, then it is not the case that $y\ P\ x$ (written not-$y\ P\ x$).

It is clearly irrational for a decision maker to pay some money to trade y for x *and*, at the same time, be willing to pay additional money to trade x for y. Note that since asymmetry implies that strict preference relation is irreflexive: for all x, it is not the case that $x\ P\ x$.

Example

Examples of asymmetric strict preferences on the set $X = \{a, b, c\}$ include:

1. $\{(a, b), (a, c)\}$: a is strictly preferred to b and strictly preferred to c, but there is no strict preference one way or the other between b and c.
2. $\{(a, b), (b, c), (a, c)\}$: a is strictly preferred to b and strictly preferred to c, and b is strictly preferred to c.
3. $\{(a, b), (b, c)\}$: a is strictly preferred to b and b is strictly preferred to c, but there is no strict preference one way or the other between a and c.
4. $\{(a, b), (b, c), (c, a)\}$: a is strictly preferred to b, b is strictly preferred to c, and c is strictly preferred to a.

In the first example, the decision maker does not have a preference over all the elements of X. In particular, the decision maker does not have a strict preference one way or the other between b and c. That is, it is not the case that bPc and it is not the case that cPb.

Definition 5.1 (No Strict Preference)**.** Suppose that P is an asymmetric relation on X. Define a relation $\simeq \subseteq X \times X$ as follows: For all $x, y \in X$,

$$x \simeq y \text{ if and only if not-}xPy \text{ and not-}yPx.$$

It is not hard to see that for any asymmetric relation P on X,

1. exactly one of $x\ P\ y$, $y\ P\ x$, and $x \simeq y$ is true.
2. $\simeq$ is symmetric for all $x, y \in X$, if $x \simeq y$ then $y \simeq x$; and
3. $\simeq$ is reflexive: for all $x \in X$, $x \simeq x$.

In many situations, it is convenient to decompose the $\simeq$ relation further. Given a strict preference P on X for a decision maker and items $x, y \in X$, there are two reasons why $x \simeq y$:

1. The decision maker is *indifferent* between x and y. In this case, we write $x\ I\ y$.
2. The decision maker *cannot compare* x and y. In this case, we write $x\ N\ y$.

Assumption 3 Suppose that $I \subseteq X \times X$ represents the decision maker's indifferences and $N \subseteq X \times X$ represents the decision maker's non-comparabilities. We assume that I is reflexive and symmetric, and that N is symmetric.

Notation

There is no settled notation for strict preferences and indifference. In some texts, you might see $\succ$ instead of P representing a strict preference and $\sim$ instead of I representing an indifference relation.

Putting everything together, a decision maker's preferences on X is represented by three relations $P \subseteq X \times X$, $I \subseteq X \times X$ and $N \subseteq X \times X$ satisfying the following minimal constraints:

1. For all $x, y \in X$, exactly one of $x\ P\ y$, $y\ P\ x$, $x\ I\ y$ and $x\ N\ y$ is true.
2. P is asymmetric
3. I is reflexive and symmetric.
4. N is symmetric.

5.2 Exercises

1. Suppose that $P \subseteq X \times X$ is an asymmetric relation and $\simeq$ as defined in Definition 5.1.
   a. Explain why $\simeq$ is symmetric.

   b. Explain why $\simeq$ is reflexive.
2. Suppose that $X = \{a, b, c\}$. Give the relations that represent the following decision makers:
   a. The decision maker strictly prefers a to b, strictly prefers b to c, and striclty prefers a to c.
   b. The decision maker strictly prefers a to b, strictly prefers b to c, and striclty prefers c to b.
   c. The decision maker strictly prefers a to b, strictly prefers b to c, and is indifferent between a and b.
   d. The decision maker strictly prefers a to b, strictly prefers b to c, and cannot compare a and b.
   e. The decision maker strictly prefers a to b, is inidfferent between b and c, and cannot compare a and c.

Chapter 6

Transitivity

A relation $R \subseteq X \times X$ is **transitive** when for all $x, y, z \in X$, if $x \mathrel{R} y$ and $y \mathrel{R} z$, then $x \mathrel{R} z$.

An key assumption in many rational choice models is that the decision maker's preference on a set X is transitive. That is, the assumption that a decision maker's preferences are transitive means that:

1. the decision maker's strict preference relation P is transitive,
2. the decision maker's indifference relation I is transitive, and
3. the decision maker's non-comparability relation N is transitive.

6.1 Transitivity of Indifference and Non-Comparability

There are reasons to reject the assumption that the decision maker's indifference relation and non-comparability relation are transitive.

Example 6.1 (Transitivity of Indifference)**.** Suppose that you are indifferent between a curry with x amount of cayenne pepper, and a curry with x plus one particle of cayenne pepper for any amount x. That is, if x is the amount of cayenne pepper in the curry, then we have that $x \mathrel{I} (x+1)$, where x is the amount of cayenne pepper added to the curry and $x+1$ represents adding x plus 1 additional particle of cayenne pepper to the curry. In particular, you have the following preferences:

$$0 \mathrel{I} 1 \qquad \text{and} \qquad 1 \mathrel{I} 2$$

Then, assuming that I is transitive, we reason as follows:

1. $0 \mathrel{I} 1$ and $1 \mathrel{I} 2$.
2. Assuming I is transitive implies that $0 \mathrel{I} 2$.
3. Given item 2 and $2 \mathrel{I} 3$, assuming that I is transitive implies that $0 \mathrel{I} 3$
4. Given item 3 and $3 \mathrel{I} 4$, assuming that I is transitive implies $0 \mathrel{I} 4$
5. And so on…

This implies that for any number x of particles of cayenne peper, you have the preference $0 \mathrel{I} x$. But you are not indifferent between a curry with no cayenne pepper and a curry with 1 pound of cayenne pepper in it!

Example 6.2 (Transitivity of Non-Comparability)**.** Suppose that you cannot compare having a job as a professor with having a job as a programmer. Furthermore, you cannot compare having a job as a programmer with having a job as a professor with an extra \$1,000. More formally, let p mean that you have a job as a professor, c mean you have a job as a programmer, and $\langle p, \$1000\rangle$ mean you have a job as a professor plus you have an extra \$1,000. Then, you have the following preferences:

$$p\ N\ c \qquad \text{and} \qquad c\ N\ \langle p, \$1000\rangle.$$

Assuming that N is transitive implies that $p\ N\ \langle p, \$1000\rangle$. However, you do strictly prefer having a job as a professor with an extra \$1,000 to having a job as a professor!

Setting aside the issues raised in Example 6.1 and Example 6.2, we assume the following:

Transitivity of Indifference and Non-Comparability Suppose that $I \subseteq X \times X$ represents a decision maker's indifference relation and that $N \subseteq X \times X$ represents a decision maker's non-comparability relation. We assume that I and N are both transitive.

- For all $x, y, z \in X$, if $x\ I\ y$ and $y\ I\ z$, then $x\ I\ z$.
- For all $x, y, z \in X$, if $x\ N\ y$ and $y\ N\ z$, then $x\ N\ z$.

6.2 Transitivity of Strict Preferences

While there are some experiments that raise doubts about whether transitivity is a good description of people's strict preferences[1], it is common to assume that a decision maker's strict preference is transitive.

There are two ways that a decision maker's strict preference P on X may fail transitivity:

1. The decision maker lacks a strict preference: There are $x, y, z \in X$ such that $x\ P\ y$ and $y\ P\ z$, but $x\ N\ z$ (i.e., x and z are incomparable).
2. There is a cycle in the decision maker's preferences: There are $x, y, z \in X$ such that $x\ P\ y$, $y\ P\ z$, and $z\ P\ x$.

To justify the assumption that a strict preference relation is transitive, we need to argue that there is something irrational about both of the above situations. In the next section, we explain how to rule out cycles in a decision maker's strict preferences. The first situation is ruled out with an additional assumption about a decision maker's preferences (see Chapter 7).

6.2.1 Ruling out Cycles

A **cycle** (of length 3) in a relation $P \subseteq X \times X$ is a sequence (x, y, z) where xPy, yPz and zPx (recall Definition 2.3). There are two main arguments that rule out preferences with cycles.

Argument 1 We cannot make sense of a decision maker with a strict preference that has a cycle. This argument is nicely expressed with the following quote from Donald Davidson:

> I do not think we can clearly say what should convince us that a [person] at a given time (without change of mind) preferred a to b, b to c and c to a. The reason for our difficulty is that we cannot make good sense of an attribution of preference except against a background of coherent attitudes…My point is that if we are intelligibly to attribute attitudes and beliefs, or usefully to describe motions as behaviour, then we are committed to finding, in the pattern of behaviour, belief, and desire, a large degree of rationality and consistency. (Davidson 2001, 237)

Argument 2: The Money-Pump Argument The Money-Pump Argument is a thought experiment demonstrating that a decision maker with a cycle in her strict preferences can end up paying an indefinite amount of money without gaining anything new. For an item $x \in X$, we write $\langle x, \$u\rangle$ to mean that the

[1]See A. Tversky's classic paper Intransitivity of Preferences and M. Regenwetter, J. Dana, and C. P. Davis-Stober, Transitivity of Preferences for a critique of these experiments.

decision maker has x and $\$u$ and write $\langle x, -\$u\rangle$ to mean that the decision maker has x and pays $\$u$. There are three key assumptions about a decision maker's strict preference P and the decision maker's opinion about money:

1. If $x \; P \; y$, then the decision maker will always take x when y is the only alternative.
2. If $x \; P \; y$, then there is some $v > 0$ such that for all u, $\langle x, -\$u\rangle \; P \; y$ if and only if $0 \leq u \leq v$.

3. The items and money are *separable* and the decision maker prefers more money to less: For all $x \in X$ and w, z, we have that $\langle x, \$w\rangle \; P \; \langle x, \$z\rangle$ if and only if $w > z$; and, for all $x, y \in X$ and w, if $x \; P \; y$, then $\langle x, \$w\rangle \; P \; \langle y, \$w\rangle$.

Suppose that Ann has a cycle in her strict preferences over the set $\{r, w, b\}$: rPw, wPb, and bPr. Furthermore, in line with assumption 2, assume that Ann is willing to pay \$1 to swap w for r, \$1 to swap b for w, and \$1 to swap r for b. That is, Ann has the following strict preferences:

$$\langle r, -\$1\rangle \; P \; w \qquad \langle w, -\$1\rangle \; P \; b \qquad \langle b, -\$1\rangle \; P \; r.$$

Suppose that Ann currently has item b. Given assumptions 1-3, we argue as follows:

1. Since $\langle b, -\$1\rangle \; P \; r$, by assumption 1, she will accept an offer to trade r for b plus she must pay \$1. After the trade, she has b and has paid \$1.

2. Now, suppose she is offered a chance to trade w for b plus she must pay \$1. Since $\langle w - \$1\rangle \; P \; b$, by assumption 1, she will accept that offer. So, she now has w and has paid \$2.
3. Suppose she is offered a chance to trade w for r plus she must pay \$1. Since $\langle r, -\$1\rangle \; P \; w$, by assumption 1, she will accept that offer. Now she has r and has paid \$3.

But she started with r and paying \$0 and ended up with r and paying \$3! By assumption 3, this is a strictly worse situation for Ann: $\langle r, \$0\rangle \; P \; \langle r, -\$3\rangle$. But it does not end here, Ann will continue to accept the offers resulting in her paying an indefinite amount of money. Ann can avoid such a money-pump argument by ensuring that there are no cycles in her strict preferences.

6.3 Exercises

1. Suppose that $X = \{a, b, c, d\}$. Which of the following relations are transitive? If the relation is not transitive, explain why.

   a. $R = \{(a, b)\}$
   b. $R = \{(a, b), (c, b), (b, a)\}$
   c. $R = \{(a, b), (b, c), (a, c)\}$
   d. $R = \{(a, b), (b, a), (a, a), (b, b), (a, c)\}$
   e. $R = \{(a, b), (b, a), (a, a), (b, b), (a, c), (b, c)\}$
   f. $R = \{(a, b), (b, c), (a, c), (c, d)\}$
   g. $R = \{(a, b), (b, c)\}$
   h. $R = \{(a, b), (b, a), (a, a)\}$

2. True or False: The Money-Pump argument shows that a rational decision maker's strict preferences must be transitive.

Solutions

1. Suppose that $X = \{a, b, c, d\}$. Which of the following relations are transitive? If the relation is not transitive, explain why.

a. $R = \{(a, b)\}$
   This relation is transitive.
b. $R = \{(a, b), (c, b), (b, a)\}$
   This relation is not transitive since $(a, b) \in R$, $(b, a) \in R$ but $(a, a) \notin R$.
c. $R = \{(a, b), (b, c), (a, c)\}$
   This relation is transitive.
d. $R = \{(a, b), (b, a), (a, a), (b, b), (a, c)\}$
   This relation is not transitive since $(b, a) \in R$, $(a, c) \in R$ but $(b, c) \notin R$.
e. $R = \{(a, b), (b, a), (a, a), (b, b), (a, c), (b, c)\}$. This relation is transitive.
f. $R = \{(a, b), (b, c), (a, c), (c, d)\}$
   This relation is not transitive since $(a, c) \in R$, $(c, d) \in R$ but $(a, d) \notin R$.
g. $R = \{(a, b), (b, c)\}$
   This relation is not transitive since $(a, b) \in R$, $(b, c) \in R$ but $(a, c) \notin R$
h. $R = \{(a, b), (b, a), (a, a)\}$
   This relation is not transitive since $(b, a) \in R$, $(a, b) \in R$ but $(b, b) \notin R$

2. True or False: The Money-Pump argument shows that a rational decision maker's strict preferences must be transitive.
   This is false. The Money-Pump argument shows that a rational decision maker's strict preferences cannot contain a cycle. We need an additional assumption to rule out situation in which a decision maker strictly prefers x to y and y to z, but cannot compare x and z.

Chapter 7

Completeness

Another key assumption in a rational choice model is that decision makers are opinionated about *all* the objects of choice. That is, there are no objects x and y such that x and y are incomparable for the decision maker.

Completeness For all $x, y \in X$, exactly one of $x\ P\ y$, $y\ P\ x$ or $x\ I\ y$ is true. I.e., for all $x, y \in X$, not-$x\ N\ y$.

Completeness is a common simplifying assumption in many rational choice models. However, assuming completeness does not have the same type of justification as transitivity:

> [O]f all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others, it is inaccurate as a description of real life; but unlike them we find it hard to accept even from the normative viewpoint. (Aumann 1962, 446)

Chapter 8

Rational Preferences

We start with some definitions:

Definition 8.1 (Strict weak order)**.** A strict weak order on a set X is a transitive and asymmetric relation on X.

Definition 8.2 (Equivalence relation)**.** An equivalence relation on X is a reflexive, transitive and symmetric relation on X.

In a rational choice model, a standard assumption is that a decision maker's preferences satisfies completeness (see Chapter 7) and transitivity (see Chapter 6).

Definition 8.3 (Rational preference)**.** A rational preference on a set X is a pair (P, I) where P is a strict weak order on X, I is an equivalence relation on X, and completeness is satisfied: for all $x, y \in X$, exactly one of $x \ P \ y$, $y \ P \ x$ or $x \ I \ y$ is true.

Note that for a rational preference (P, I), there is no $x, y \in X$ such that $x \ P \ y$ and $x \ I \ y$ (i.e., a rational agent cannot both strictly prefer x to y and be indifferent between x and y). Using this fact and transitivity of P and I, we have the following two key properties:

1. For all $x, y, z \in X$, if $x \ P \ y$ and $y \ I \ z$, then $x \ P \ z$.
2. For all $x, y, z \in X$, if $x \ I \ y$ and $y \ P \ z$, then $x \ P \ z$.

In many situations it is natural to assume that the decision maker is not indifferent about any of the items in a set X (i.e., the decision maker's indifference relation is empty). In such a case, we can represent a decision maker by a single relation P (the decision maker's strict preference relation) satisfying the following properties:

Definition 8.4 (Linear order)**.** A linear order on a set X is a transitive, connected and asymmetric relation on X.

https://www.youtube.com/embed/XO_YbpPNrRw?si=ApT5MlZLmV05e7hS

8.1 Weak preference relation

> ⚠ Warning
>
> This section contains more advanced material and can be skipped on a first reading.

Even if a decision maker is indifferent between some items, we can represent the decision maker's rational preferences by a single relation.

Definition 8.5 (Derived weak preference relation)**.** Suppose that (P, I) is a rational preference on X. The **weak preference relation based on** (P, I) is defined as follows: $R \subseteq X \times X$, where $x\ R\ y$ if and only if $x\ P\ y$ or $x\ I\ y$. If $x\ R\ y$, we say that "x is weakly preferred to y".

It is not hard to see that if R is a weak preference relation based on a rational preference (P, I), then R is a reflexive, transitive and connected relation.

We can also start with a weak preference relation and induce strict preference and an indifference relation.

Definition 8.6 (Rational weak preference)**.** Suppose that X is a set. A rational weak preference on X is a relation $R \subseteq X \times X$ that is reflexive, transitive and connected.

A key observation is that rational weak preferences can be used to represent a decision maker's rational preferences.

Lemma 8.1. *Suppose that $R \subseteq X \times X$ is a reflexive and transitive relation. Define relations $P_R \subseteq X \times X$ and $I_R \subseteq X \times X$ as follows:*

- *$x\ P_R\ y$ if and only if $x\ R\ y$ and not-$y\ R\ x$.*
- *$x\ I_R\ y$ if and only if $x\ R\ y$ and $y\ R\ x$.*

Then, (P_R, I_R) is a rational preference on X.

Proof. Clearly, there are no $x, y \in X$ such that $x\ P_R\ y$ and $x\ I_R\ y$.

We prove that P_R is transitive: Suppose that $x\ P_R\ y$ and $y\ P_R\ z$. Then $x\ R\ y$, not-$y\ R\ x$, $y\ R\ z$ and not-$z\ R\ y$. Since $x\ R\ y$, $y\ R\ z$, and R is transitive, we have that $x\ R\ z$. To prove that $x\ P_R\ z$ we must show that not-$z\ R\ x$. Towards a contradiction, suppose that $z\ R\ x$. Then since $z\ R\ x$, $x\ R\ y$, and R is transitive, we have that $z\ R\ y$, which contradicts the assumption that not-$z\ R\ y$. Hence, $x\ R\ z$ and not-$z\ R\ x$, and so $x\ P_R\ z$. The proof that P_R is asymmetric is left as an exercise.

The proof that I_R is an equivalence relation is left as an exercise.

Finally, we show that (P_R, I_R) satisfies completeness. Suppose that $x, y \in X$. Since R is connected, we have either $x\ R\ y$ or $y\ R\ x$. There are three possibilities:

1. $x\ R\ y$ and not-$y\ R\ x$. In this case, $x\ P_R\ y$.
2. not-$x\ R\ y$ and $y\ R\ x$. In this case, $y\ P_R\ x$.
3. $x\ R\ y$ and $y\ R\ x$. In this case, $x\ I_R\ y$.

□

8.2 Exercises

1. Suppose (P, I) is a rational preference on X. Explain why the following are true:

   a. For all $x, y, z \in X$, if $x\ P\ y$ and $y\ I\ z$, then $x\ P\ z$.

b. For all $x, y, z \in X$, if $x \; I \; y$ and $y \; P \; z$, then $x \; P \; z$.

2. Suppose that $X = \{a, b, c, d\}$ and that

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (a, d), (b, c), (c, b), (b, d), (c, d)\}$$

represents a decision maker's weak preference over the items in X. Select all the statements that are true about the decision maker:

a. The decision maker strictly prefers a over b.
b. The decision maker strictly prefers b over a.
c. The decision maker is indifferent between a and b.
d. The decision maker strictly prefers a over c.
e. The decision maker strictly prefers c over a.
f. The decision maker is indifferent between a and c.
g. The decision maker strictly prefers b over c.
h. The decision maker strictly prefers c over b.
i. The decision maker is indifferent between b and c.

3. Suppose that (P, I) is a rational preference on X and R is the derived weak preference relation (see Definition 8.6). Suppose that $x, y \in X$ and $x \; R \; y$ and $y \; P \; z$. Which of the following is true:

a. $x \; P \; z$
b. $z \; P \; x$
c. $x \; R \; z$
d. $z \; R \; x$

4. Suppose that Ann's preferences are rational and that Ann strictly prefers a over b and she strictly prefers b over c. What can you conclude about Ann's preference of a and c?

5. Suppose that Ann's preferences are rational and that Ann strictly prefers a over b and she strictly prefers b over c. What can you conclude about Ann's preference of a and c?

a. Ann strictly prefers a over c.
b. Ann strictly prefers c over a.
c. Ann is indifferent between a and c.
d. Ann cannot compare a and c.
e. There is not enough information to answer this question.

6. Suppose that Ann's preferences are rational and that Ann strictly prefers a over c and she strictly prefers b over c. What can you conclude about Ann's preference of a and b?

a. Ann strictly prefers a over b.
b. Ann strictly prefers b over a.
c. Ann is indifferent between a and b.
d. Ann cannot compare a and b.
e. There is not enough information to answer this question.

7. Suppose that Ann's preferences are rational, Ann strictly prefers a over b, and c is some alternative different from both a and b. What else can you conclude about Ann's preferences?

a. Ann strictly prefers a over c and she strictly prefers c over b.

b. Ann strictly prefers a over c or she strictly prefers c over b.

c. There is not enough information to answer this question.

Solutions

1. Suppose (P, I) is a rational preference on X. Explain why the following are true:
   a. For all $x, y, z \in X$, if $x\ P\ y$ and $y\ I\ z$, then $x\ P\ z$.
   Suppose that (P, I) is a rational preference and for $x, y, z \in X$ we have that $x\ P\ y$ and $y\ I\ z$. By completeness, exactly one of $x\ P\ z$, $z\ P\ x$ or $x\ I\ z$ is true. We show that both $z\ P\ x$ and $x\ I\ z$ lead to a contradiction leaving only $x\ P\ z$, as desired. If $z\ P\ x$, then since $x\ P\ y$ and P is transitive, we have that $z\ P\ y$. Since I is symmetric and $y\ I\ z$, we have that $z\ I\ y$. This is a contradiction since $z\ P\ y$ and $z\ I\ y$ cannot both be true. If $z\ I\ x$, then since $y\ I\ z$ and I is transitive, we have that $y\ I\ x$. Since I is symmetric, we have that $x\ I\ y$. This is a contradiction since $x\ P\ y$ and $x\ I\ y$ cannot both be true.
   b. For all $x, y, z \in X$, if $x\ I\ y$ and $y\ P\ z$, then $x\ P\ z$.
   Suppose that (P, I) is a rational preference and for $x, y, z \in X$ we have that $x\ I\ y$ and $y\ P\ z$. By completeness, exactly one of $x\ P\ z$, $z\ P\ x$ or $x\ I\ z$ is true. We show that both $z\ P\ x$ and $x\ I\ z$ lead to a contradiction leaving only $x\ P\ z$, as desired. If $z\ P\ x$, then since $y\ P\ z$ and P is transitive, we have that $y\ P\ x$. Since I is symmetric and $x\ I\ y$, we have that $y\ I\ x$. This is a contradiction since $y\ P\ x$ and $y\ I\ x$ cannot both be true. If $z\ I\ x$, then since $x\ I\ y$ and I is transitive, we have that $z\ I\ y$. Since I is symmetric, we have that $y\ I\ z$. This is a contradiction since $y\ P\ z$ and $y\ I\ z$ cannot both be true.
2. Suppose that $X = \{a, b, c, d\}$ and that

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (a, d), (b, c), (c, b), (b, d), (c, d)\}$$

   represents a decision maker's weak preference over the items in X. Select all the statements that are true about the decision maker:
   a. The decision maker strictly prefers a over b: This is true since $(a, b) \in R$ and $(b, a) \notin R$.
   b. The decision maker does not strictly prefer b over a: This is not true since $(b, a) \notin R$ (but we do have $(a, b) \in R$).
   c. The decision maker is not indifferent between a and b: This is not true since $(b, a) \notin R$ (but we do have $(a, b) \in R$).
   d. The decision maker strictly prefers a over c: This is true since $(a, c) \in R$ and $(c, a) \notin R$.
   e. The decision maker strictly prefers c over a: This is not true since $(c, a) \notin R$ (but we do have $(a, c) \in R$).
   f. The decision maker is not indifferent between a and c: This is not true since $(c, a) \notin R$ (but we do have $(a, c) \in R$).
   g. The decision maker does not strictly prefer b over c: This is not true since $(b, c) \in R$ and $(c, b) \in R$.
   h. The decision maker does not strictly prefer c over b: This is not true since $(b, c) \in R$ and $(c, b) \in R$.
   i. The decision maker is indifferent between b and c: This is true since $(b, c) \in R$ and $(c, b) \in R$
3. Suppose that (P, I) is a rational preference on X and R is the derived weak preference relation (see Definition 8.6). Suppose that $x, y \in X$ and $x\ R\ y$ and $y\ P\ z$. Which of the following is true:
   a. (✓) $x\ P\ z$
   b. $z\ P\ x$
   c. $x\ R\ z$
   d. $z\ R\ x$

   We must have $x\ P\ z$.
   Since $x\ P\ y$, we have either $x\ P\ y$ or $x\ I\ y$. If $x\ P\ y$, then by transitivity of P, we have that $x\ P\ z$. If $x\ I\ y$, then using the argument given in the answer to question A., we must have that $x\ P\ z$. In both cases, xPz.

4. Suppose that Ann's preferences are rational and that Ann strictly prefers a over b and she strictly prefers b over c. What can you conclude about Ann's preference of a and c?
   Ann strictly prefers a over c.
   Let (P, I) be Ann's rational preferences. Since P is transitive and we have that aPb and bPc, then aPc.
5. Suppose that Ann's preferences are rational and that Ann strictly prefers a over b and she strictly prefers b over c. What can you conclude about Ann's preference of a and c?
   a. (✓) Ann strictly prefers a over c.
   b. Ann strictly prefers c over a.
   c. Ann is indifferent between a and c.
   d. Ann cannot compare a and c.
   e. There is not enough information to answer this question.

   Ann strictly prefers a over c.
   Let (P, I) be Ann's rational preferences. Since P is transitive and we have that aPb and bPc, then aPc.
6. Suppose that Ann's preferences are rational and that Ann strictly prefers a over c and she strictly prefers b over c. What can you conclude about Ann's preference of a and b?
   a. Ann strictly prefers a over b.
   b. Ann strictly prefers b over a.
   c. Ann is indifferent between a and b.
   d. Ann cannot compare a and b.
   e. (✓) There is not enough information to answer this question.

   There is not enough information to answer this question.
7. Suppose that Ann's preferences are rational, Ann strictly prefers a over b, and c is some alternative different from both a and b. What else can you conclude about Ann's preferences?
   a. Ann strictly prefers a over c and she strictly prefers c over b.
   b. (✓) Ann strictly prefers a over c or she strictly prefers c over b.
   c. There is not enough information to answer this question.

   Ann striclty prefers a over c or she strictly prefers c over b.
   Suppose that (P, I) represents Ann's rational preferences. By completeness, we have either aPc, cPa, or aIc. Also by completeness, we have either bPc, cPb, or bIc. This gives a total of 9 possible situations. The following list describes all the possible situations that can arise and what they imply about Ann's preference between a and c and between c and b:
   1. aPc and bPc: In this case we have that bPc
   2. aPc and cPb: In this case we have that aPc and cPb
   3. aPc and bIc: In this case we have that aPc
   4. cPa and bPc: This is impossible since this implies that bPa
   5. cPa and cPb: In this case we have that cPb
   6. cPa and bIc: This is impossible since this implies that bPa
   7. aIc and bPc: This is impossible since this implies that bPa
   8. aIc and cPb: In this case we have that cPb
   9. aIc and bIc: This is impossible since this implies that bPa

   The only possibilities are that either Ann strictly prefers a over c or she strictly prefers c over b.

Chapter 9

Maximal Elements

This section studies the relationship between preference and choice. The standard assumption in rational choice models is that a decision maker will choose an element from a set of feasible alternatives A that is "best" according to her preference. There are two ways to define the "best" element of a set with respect to some relation R on that set.

Definition 9.1 (Maximum)**.** Suppose that X is a set, $R \subseteq X \times X$ is a relation on X, and $A \subseteq X$. We say that $x \in A$ is a **maximum** element of A with respect to R provided that

$$\text{for all } y \in A,\ x \mathrel{R} y.$$

Definition 9.2 (Maximal)**.** Suppose that X is a set, $R \subseteq X \times X$ is a relation on X, and $A \subseteq X$. We say that $x \in A$ is a a **maximal** element of A with respect to R provided that

$$\text{there is no } y \in A \text{ such that } y \mathrel{R} x.$$

The following examples illustrate the above definition: Suppose that $X = \{a, b, c\}$.

- Let $R = \{(a, b), (b, c), (a, c)\}$ and $A = \{b, c\}$. Then,
  - a is the only maximal element of X with respect to R.
  - b is the only maximal element of A with respect to R.
  - a is the only maximum element of X with respect to R.
  - b is the only maximum element of A with respect to R.
- Let $R = \{(a, b), (b, c), (c, a)\}$ and $A = \{a, c\}$. Then,
  - There are no maximal elements of X with respect to R.
  - c is the only maximal element of A with respect to R.
  - There is no maximum element of X with respect to R.
  - c is the only maximum element of A with respect to R.
- Let $R = \{(a, b), (c, b)\}$. Then,
  - a and c are both maximal elements of X with respect to R.
  - There is no maximum element of X with respect to R.

https://www.youtube.com/embed/JYjZSNzp9EY?si=cK6mMfP01XfqO5AU

9.1 Rational Choice

Suppose that X is a set of alternatives and $A \subseteq X$. We write $C(A)$ for the set of *admissible*, or *choice-worthy*, elements of A for a decision maker. The interpretation is that when a decision maker must choose an alternative from a set A of feasible options, she will pick something from the set $C(A) \subseteq A$.

A decision maker's choice is **rational** when there is a rational preference (P, I) on X such that for all $A \subseteq X$, the set $C(A)$ of choice-worthy elements of A is the set of maximal elements of A with respect to her strict preference P.

> **i Actual vs. Hypothetical Choices**
>
> The mathematical formalism does not specify whether a choice function C represents a decision maker's *actual* or *hypothetical* choice. If it is the actual choices, then C is a record of the decision maker's observed choice behavior. If it is the hypothetical choices, then C represents what the decision maker *would* chose if given the opportunity to select an element from a given menu.

9.1.1 Revealed Preference Theory

> **⚠ Warning**
>
> This section contains more advanced material and can be skipped on a first reading.

When a decision maker uniquely choses x from a set containing x and y (i.e., $C(\{x, y\}) = \{x\}$), we say that she *reveals a preference* for x over y. There is a elegant mathematical theory identifying the preferences revealed by the choices of a decision maker.

> Standard economics focuses on revealed preference because economic data comes in this form. Economic data can—at best—reveal what the agent wants (or has chosen) in a particular situation. Such data do not enable the economist to distinguish between what the agent intended to choose and what he ended up choosing; what he chose and what he ought to have chosen. (Gul and Pesendorfer 2008)

Definition 9.3 (Derived Choice Function)**.** Suppose that R is a relation on a finite set X. The choice function derived from the relation R is $C_R : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: for all $A \in \mathcal{P}(X) \ \ \emptyset$,

$$C_R(A) = \{y \mid y \in A \text{ and there is no } x \in A \text{ such that } x \mathrel{R} y\}.$$

Definition Definition 9.3 can be applied to *any* relation on a set X. In general, given an arbitrary relation R on X, C_R may not necessarily be a *choice* function. This would happen when there is a finite subset $Y \subseteq X$ such that $C_R(Y) = \emptyset$. The following Lemma states precisely when a function derived from a relation is a choice function.

Lemma 9.1. *Suppose that X is finite. A binary relation $R \subseteq X \times X$ is acyclic if and only if C_R is a choice function.*

Proof. Suppose that $R \subseteq X \times X$ is acyclic. By definition, for any nonempty set $S \in \mathcal{P}(X)$, $C_R(S) \subseteq S$. We must show $C_R(S) \neq \emptyset$. Suppose that $C_R(S) = \emptyset$. Choose an element $x_0 \in S$. Since $C_R(S) = \emptyset$, there is an element $x_1 \in S$ such that $x_1 \mathrel{R} x$. Again, since $C_R(S) = \emptyset$ there must be some element $x_2 \in S$ such that $x_2 \mathrel{R} x_1$. Since R is acyclic, we must have $x_2 \neq x$ (otherwise, $x \mathrel{R} x_1 \mathrel{R} x$ is a cycle). Continue in this manner selecting elements of S. Since S is finite, eventually all elements of S are selected. That is, we have $S = \{x_0, x_1, x_2, \dots, x_n\}$ and

$$x_n \mathrel{R} x_{n-1} \mathrel{R} \cdots x_2 \mathrel{R} x_1 \mathrel{R} x_0$$

Since $C_R(S) = \emptyset$ there must be some element $x \in S$ such that $x \mathrel{R} x_n$. Thus, $x = x_i$ for some $i = 0, \dots, n$, which implies R has a cycle. This contradicts the assumption that $C_R(S) = \emptyset$. Hence $C_R(S) \neq \emptyset$.

Suppose that C_R is a choice function. This means that for all $S \in \mathcal{P}(X)$, $C_R(S) \neq \emptyset$. Suppose that R is not acyclic. Then, there is a set of distinct elements $x_1, x_2, \dots, x_n \in S$ such that

$$x_1 \; R \; x_2 \; R \cdots x_{n-1} \; R \; x_n \; R \; x_1.$$

But this means that $C_R(\{x_1, \dots, x_n\}) = \emptyset$. (The above cycle means that there is no maximal element of $\{x_1, \dots, x_n\}$.) This contradicts the assumption that C_R is a choice function. Thus, R is acyclic.

Suppose that (P, I) are the rational preference on X for a decision maker. Since P is acyclic, by Lemma 1, C_P is a choice function. A choice function represents the choices of a decision maker when there is some rational preference that generates the choices. □

□

Definition 9.4 (Rationalizable Choice Functions)**.** A choice function $C : \mathcal{P}(X) \to \mathcal{P}(X)$ is **rationalizable** if there is a rational preference (P, I) on X such that for all $A \in \mathcal{P}$, $C(A) = C_P(A)$.

Not every choice function is rationalizable. There are two key properties that completely characterize rationalizable choice functions. Suppose that C is a choice function on X. We say that C satisfies:

1. **Sen's property** provided that for all $A, B \subseteq X$, for all $x \in X$, if $x \in B \subseteq A$ and $x \in C(A)$, then $x \in C(B)$
2. **Sen's property** provided that for all $A, B, \subseteq X$, for all $x, y \in X$ if $x, y \in C(A)$, $A \subseteq B$ and $y \in C(B)$, then $x \in C(B)$.

The first result is that Sen's α and β are together equivalent to a single axiom knowns as the *weak axiom of revealed preference* (WARP). Suppose that C is a choice function on X. We say that C satisfies:

3. **Weak Axiom of Revealed Preference** (also known as WARP or Houthakker's Axiom) provided that for all $A, B, \subseteq X$, for all $x, y \in X$, if x and y are both contained in A and B and if $x \in C(A)$ and $y \in C(B)$ then $x \in C(B)$.

Lemma 9.2. *A choice function C satisfies WARP if and only if C satisfies Sen's properties α and β.*

Proof. Suppose that C satisfies WARP. We must show C satisfies Sen's α and β:

- C satisfies Sen's α: Suppose that $x \in X$ with $x \in B \subseteq A \subseteq X$ and $x \in C(A)$. Suppose that $x \notin C(B)$. Then there is some $y \in B$ such that $y \in C(B)$ and $y \neq x$. Since $y \in B$ and $B \subseteq A$, we have $y \in A$. Hence, x and y are in both A and B. By the WARP axiom, since $x \in C(A)$ and $y \in C(B)$, we must have $x \in C(B)$. This contradicts the assumption that $x \notin C(B)$. Thus, $x \in c(B)$, as desired.
- C satisfies Sen's β: Suppose that $x, y \in X$ with $x, y \in C(A)$, $A \subseteq B \subseteq X$ and $y \in C(B)$. Since $C(A) \subseteq A$, we have $x, y \in A$; and since $A \subseteq B$, we have $x, y \in B$. Thus, x and y are in both A and B. By the WARP axiom, since $x \in C(A)$ and $y \in C(B)$, we must have $x \in C(B)$, as desired.

Now, suppose that C satisfies Sen's α and β. We must show that C satisfies WARP. Suppose that $x, y \in A \cap B$, $x \in C(A)$ and $y \in C(B)$. We must show that $x \in C(B)$. Since, $A \cap B \subseteq B$ and $y \in C(B)$, by Sen's α, $y \in C(A \cap B)$. Similarly, since $A \cap B \subseteq A$ and $x \in C(A)$, by Sen's α, $x \in C(A \cap B)$. Finally, Since $x, y \in C(A \cap B)$, $A \cap B \subseteq B$ and $y \in C(B)$, by Sen's β, we have that $x \in C(B)$, as desired.

□

The main result of this section is that WARP is equivalent to rationalizability.

Theorem 9.1. *Suppose that X is a finite set and C is a choice function on X. Then, C satisfies WARP if and only if C is rationalizable.*

Proof. Suppose C is a choice function on X, and that C is rationalizable. Then there is a rational preference (P, I) such that $C = C_P$. We must show that C satisfies WARP. Suppose that $A, B \subseteq X$ and $x, y \in A \cap B$, $x \in C(A)$ and $y \in C(B)$. We must show that $x \in C(B)$. Since $C = C_P$, we have that x is a maximal element in A with respect to P and that y is a maximal element in B with respect to P. This means that there is no $z \in A$ such that zPx and there is no $z \in B$ such that zPy. Suppose that $w \in B$. We will show that not-wPx. Since $w \in B$ and y is maximal in B with respect to P, we have that not-wPy. Since (P, I) is complete, this means that yPw or yIw (i.e, yR_Pw). Furthermore, since $y \in A$ and x is maximal in A with respect to P, we have that not-yPx. Since (P, I) is complete, this means that either xPy or xIy (i.e., xR_Py). Since R_P is transitive and $x\ R_P\ y$ and $y\ R_P\ w$, we have that $x\ R_P\ w$. This implies that not-wPx. That is, x is maximal in B with respect to P, i.e., $x \in C(B)$.

Suppose that C satisfies WARP. Then by Lemma 2, C satisfies Sen's α and β. Define a relation $R_C \subseteq X \times X$ as follows: for all $x, y \in X$,

$$x\ R_C\ y \text{ if and only if } x \in C(\{x, y\}).$$

We must show that 1. R_C is a rational weak preference relation and 2. for all $S \in \mathcal{P}(X)$, $C(S) = C_{P_C}(S)$, where P_C is the strict preference relation derived from R_C. To see that 1. holds:

R_C is connected: For any $x, y \in X$, since $C(\{x, y\})$ is non-empty we have that $C(\{x, y\}) = \{x\}$, $C(\{x, y\}) = \{y\}$ or $C(\{x, y\}) = \{x, y\}$. Thus, either $x\ R_C\ y$ or $y\ R_C\ x$ (or both).

R_C is transitive: Suppose that $x\ R_C\ y$ and $y\ R_C\ z$. Then, $x \in C(\{x, y\})$ and $y \in C(\{y, z\})$. We must show that $x\ R_C\ z$; that is, $x \in C(\{x, z\})$. By Sen's α, if $x \in C(\{x, y, z\})$, then $x \in C(\{x, z\})$. Thus, if we show that $x \in C(\{x, y, z\})$, then we are done. There are three cases:

1. Suppose that $C(\{x, y, z\}) = \{y\}$. By Sen's α, since $\{x, y\} \subseteq \{x, y, z\}$ and $y \in C(\{x, y, z\})$ we must have $y \in C(\{x, y\})$. Thus, $C(\{x, y\}) = \{x, y\}$. By Sen's β, this implies that $x \in C(\{x, y, z\})$ (this follows since $\{x, y\} \subseteq \{x, y, z\}$, $x, y \in C(\{x, y\})$ and $y \in C(\{x, y, z\})$). This contradicts the assumption that $C(\{x, y, z\}) = \{y\}$. Thus, $C(\{x, y, z\} \neq \{y\}$.

2. A similar argument shows that $C(\{x, y, z\}) \neq \{z\}$.

3. Suppose that $C(\{x, y, z\}) = \{y, z\}$. Then, $y \in C(\{x, y, z\})$, and, as above, by Sen's α, we have $C(\{x, y\}) = \{x, y\}$. This implies, by Sen's β, that $x \in C(\{x, y, z\})$, which contradicts that assumption that $C(\{x, y, z\}) = \{y, z\}$.

Hence, $x \in C(\{x, y, z\}$. By Sen's α, since $\{x, z\} \subseteq \{x, y, z\}$, we have $x \in C(\{x, z\})$. That is, $x\ R_C\ z$. This completes the proof that R_C is transitive.

Let P_C be the strict preference relation derived from R_C. Suppose that $S \in \mathcal{P}(X)$. First of all, if S is a singleton (i.e., $S = \{x\}$ for some $x \in X$), then by definition $C(S) = S = C_{P_C}(S)$. Thus, in what follows we assume that S has at least two elements. We must show that $C(S) = C_{P_C}(S)$. We first show that $C(S) \subseteq C_{P_C}(S)$. Suppose that $x \in C(S)$. We must show that $x \in C_{P_C}(S)$. Let $y \in S$. We must show that not-$y\ P_C\ x$. Since R_C is connected, this is equivalent to showing that $x\ R_C\ y$. Since $\{x, y\} \subseteq S$ and $x \in C(S)$, by Sen's α, we have $x \in C(\{x, y\})$. Thus, $x\ R_C\ y$; and so, not-$y\ P_C\ x$, which implies that $x \in C_{P_C}(S)$. Next, we show that $C_{P_C}(S) \subseteq C(S)$. Suppose that $x \in C_{P_C}(S)$. Suppose that $x \notin C(S)$. Then there is some $y \neq x$ such that $y \in C(S)$. By Sen's α, this implies that $y \in C(\{x, y\})$. Furthermore, if $C(\{x, y\}) = \{x, y\}$, then, by Sen's β, $x \in C(S)$. This contradicts the assumption that $x \notin C(S)$. Thus, $C(\{x, y\}) = \{y\}$. By definition, this means that $y\ R_C\ x$ but not-$x\ R_C\ y$; i.e., yP_Cx. This contradicts the assumption that $x \in C_{P_C}(S)$. Thus, $x \in C(S)$, as desired.

□

9.2 Exercises

1. Suppose that $X = \{a, b, c, d\}$ and $R = \{(a, b), (b, c), (a, c)\}$. What are the set of maximal elements in $A = \{a, b, c\}$ according to R? What are the set of maximal elements in $A = \{a, b, c, d\}$ according to R?
2. Is it possible to find a relation R on X that has a cycle and such that there is a non-empty set of maximal elements of X according to R?
3. Is it possible to find a relation R on a set X that is reflexive and such that there is a non-empty set of maximal elements of X according to R?
4. Suppose that $X = \{a, b, c\}$. Consider the a decision maker that makes the following choices:
   - From $\{a, c\}$ the decision maker chooses uniquely a
   - From $\{a, b, c\}$ the decision maker chooses uniquely b

   Is there a rational preference (P, I) on X such that the decision maker chooses according to that preference?
5. Suppose that $X = \{a, b, c, d\}$. Consider the a decision maker that makes the following choices:
   - From $\{a, c\}$ the decision maker chooses uniquely a
   - From $\{a, b, c, d\}$ the decision maker chooses both a and b

   Is there a rational preference (P, I) on X such that the decision maker chooses according to that preference?
6. Suppose that $X = \{a, b, c, d\}$. Consider the a decision maker that makes the following choices:
   - From $\{a, c\}$ the decision maker chooses uniquely a
   - From $\{a, b, c, d\}$ the decision maker chooses uniquely c

   Is there a rational preference (P, I) on X such that the decision maker chooses according to that preference?
7. Suppose that $X = \{a, b, c, d\}$. Consider the a decision maker that makes the following choices:
   - From $\{a, c\}$ the decision maker chooses uniquely a
   - From $\{a, b, c, d\}$ the decision maker chooses uniquely d

   Is there a rational preference (P, I) on X such that the decision maker chooses according to that preference?

Solutions

1. Suppose that $X = \{a, b, c, d\}$ and $R = \{(a, b), (b, c), (a, c)\}$. What are the set of maximal elements in $A = \{a, b, c\}$ according to R? What are the set of maximal elements in $A = \{a, b, c, d\}$ according to R?
   The maximal element of $\{a, b, c\}$ according to R is a.
   The maximal elements of $\{a, b, c, d\}$ according to R are a and d.
2. Is it possible to find a relation that has a cycle and a non-empty set of maximal elements?
   Yes, suppose that $X = \{a, b, c, d\}$ and
   $$R = \{(a, b), (a, c), (a, d), (b, c), (c, d), (d, b)\}$$
   The maximal element of X is a according to R and there is a cycle (b, c, d) in R.
3. Is it possible to find a relation R on a set X that is reflexive and such that there is a non-empty

set of maximal elements of X according to R?
No, if R is reflexive then for all x, $x \ R \ x$, so there cannot be any maximal elements in X according to R.

4. Suppose that $X = \{a, b, c\}$. Consider the a decision maker that makes the following choices:
   - From $\{a, c\}$ the decision maker chooses uniquely a
   - From $\{a, b, c\}$ the decision maker chooses uniquely b

   Is there a rational preference (P, I) on X such that the decision maker chooses according to that preference?
   Yes, suppose that the decision maker's strict preference is $P = \{(a, c), (b, a), (b, c)\}$ and indfference is $I = \{(a, a), (b, b), (c, c)\}$
5. Suppose that $X = \{a, b, c, d\}$. Consider the a decision maker that makes the following choices:
   - From $\{a, c\}$ the decision maker chooses uniquely a
   - From $\{a, b, c, d\}$ the decision maker chooses both a and b

   Is there a rational preference (P, I) on X such that the decision maker chooses according to that preference?
   Yes, suppose that the decision maker's strict preference is $P = \{(a, c), (b, c), (c, d), (a, d), (b, d)\}$ and indfference is $I = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$.
6. Suppose that $X = \{a, b, c, d\}$. Consider the a decision maker that makes the following choices:
   - From $\{a, c\}$ the decision maker chooses uniquely a
   - From $\{a, b, c, d\}$ the decision maker chooses uniquely c

   Is there a rational preference (P, I) on X such that the decision maker chooses according to that preference?
   No, since $\{a, c\} \subseteq \{a, b, c, d\}$ and if c is maximal in $\{a, b, c, d\}$ according to the decision maker's strict preferences, then c should be maximal in $\{a, c\}$ according to the decision maker's strict preferences.
7. Suppose that $X = \{a, b, c, d\}$. Consider the a decision maker that makes the following choices:
   - From $\{a, c\}$ the decision maker chooses uniquely a
   - From $\{a, b, c, d\}$ the decision maker chooses uniquely d

   Is there a rational preference (P, I) on X such that the decision maker chooses according to that preference?
   Yes, suppose that the decision maker's strict preference is $P = \{(a, c), (d, a), (d, c), (c, b), (a, b), (d, b)\}$ and indfference is $I = \{(a, a), (b, b), (c, c)\}$

Part III

Utility

A utility function assigns numbers to alternatives called *utilities* with the intended interpretation that the alternatives with higher utilities are preferred to alternatives with lower utilities. More formally, given a set X of alternatives, a **utility function** for a decision maker is a function $u : X \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. When $u(x) = r$ we say that "the utility of x is r".

The main readings for this section are:

- Chapter 4, Section 4.1 from Hausman, McPherson, and Satz (2020)
- Chapter 2 from Gaus and Thrasher (2021)
- Chapter 3, pp. 29 - 42 from Reiss (2013)

Additional readings about utility include:

- Broome (1991) which can be downloaded here
- Chapter 2 from Gilboa (2012)

Chapter 10

Representing Preferences

The standard interpretation of a utility function in Rational Choice Theory is that it is an indicator of preference. This means that there is an important relationship between utility functions and rational preferences.

The first observation is that for any utility function $u : X \to \mathbb{R}$ on a set X, we can define relations $P_u \subseteq X \times X$ and $I_u \subseteq X \times X$ as follows: for all $x, y \in X$,

1. $x\ P_u\ y$ when $u(x) > u(y)$; and
2. $x\ I_u\ y$ when $u(x) = u(y)$.

It is not hard to see that for any utility function $u : X \to \mathbb{R}$ on a set X, (P_u, I_u) is a rational preference on X.

The second observation is that every rational preference (P, I) on a set X can be **represented** by a utility function.

Definition 10.1. Suppose that X is a set $P \subseteq X \times X$ and $I \subseteq X \times X$ are two relations. We say that (P, I) is representable when there is a function $u_{P,I} : X \to \mathbb{R}$ such that, for all $x, y \in X$:

1. if $x\ P\ y$, then $u_{P,I}(x) > u_{P,I}(y)$; and
2. if $x\ I\ y$, then $u_{P,I}(x) = u_{P,I}(y)$.

Putting these two observations together, we have that a pair of relations (P, I) are a rational preference exactly when the relations are representable.

Theorem 10.1 (Basic Representation Theorem)**.** *Suppose that X is a finite set and $P \subseteq X \times X$ and $I \subseteq X \times X$. Then, (P, I) is a rational preference on X if, and only if, (P, I) is representable by a utility function.*

Proof. We leave it to the reader to show that if (P, I) is representable by a utility function, then (P, I) is a rational preference on X. That is, for all utility functions $u : X \to \mathbb{R}$, (P_u, I_u) is a rational preference.

We prove the following: For all $n \in \mathbb{N}$, any rational preference (P, I) on a set of size n is representable by a utility function $u_{P,I} : X \to \mathbb{R}$. The proof is by induction on the size of the set of objects X. The base case is when $|X| = 1$. In this case, $X = \{a\}$ for some object a. If (P, I) is a rational preference on X, then $P = \emptyset$ and $I = \{(a, a)\}$. Then, $u_{P,I}(a) = 0$ (any real number would work here) clearly represents (P, I). The induction hypothesis is: if $|X| = n$, then any rational preference (P, I) on X is representable. Suppose that $|X| = n + 1$ and (P, I) is a rational preference on X. Then, $X = Y \cup \{a\}$ for some object a, where $|Y| = n$. Note that the *restriction* of (P, I) to Y, denoted (P_Y, I_Y) where $P_Y = P \cap (Y \times Y)$ and $I_Y = I \cap (Y \times Y)$, is a rational preference on Y. By the induction hypothesis, there is a utility function $u_{P_Y,I_Y} : Y \to \mathbb{R}$ that represents (P_Y, I_Y). We will show how to extend u_{P_Y,I_Y} to a utility function $u_{P,I} : X \to \mathbb{R}$ that represents (P, I). For all $b \in Y$, let $u_{P,I}(b) = u_{P_Y,I_Y}(b)$. For the object a (the unique object in X but not in Y), there are four cases:

1. $a\ P\ b$ for all $b \in Y$. Let $u_{P,I}(a) = \max\{u_{P_Y,I_Y}(b) \mid b \in X'\} + 1$.
2. $b\ P\ a$ for all $b \in Y$. Let $u_{P,I}(a) = \min\{u_{P_Y,I_Y}(b) \mid b \in X'\} - 1$.
3. $a\ I\ b$ for some $b \in Y$. Let $u_{P,I}(a) = u_{P_Y,I_Y}(b)$.
4. There are $b_1, b_2 \in Y$ such that $b_1\ P\ a\ P\ b_2$. Let $u_{P,I}(a) = \frac{u_{P_Y,I_Y}(b_1)+u_{P_Y,I_Y}(b_2)}{2}$.

Then, it is straightforward to show that $u_{P,I} : X \to \mathbb{R}$ represents (P, I) (the details are left to the reader).

□

The above proof can be extended to relations on infinite sets X. However, additional technical assumptions are needed, which are beyond the scope of this course.

10.1 Exercises

1. Find three utility functions that represent the rational preference relation (P, I) on $X = \{a, b, c, d\}$, where
$$P = \{(a, b), (b, c), (a, c), (d, b), (d, c)\}$$
and
$$I = \{(a, a), (b, b), (c, c), (a, d), (d, a)\}.$$

2. Suppose that (P, I) is a rational preference on X and that $u : X \to \mathbb{R}$ represents (P, I).

   a. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be the function where for all $x \in \mathbb{R}$, $f(x) = 2x + 3$. Then, $f \circ u$ represents (P, I).

   b. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be the function where for all $x \in \mathbb{R}$, $f(x) = -5x$. Then, $f \circ u$ represents (P, I).

   c. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be the function where for all $x \in \mathbb{R}$, $f(x) = x^2$. Then, $f \circ u$ represents (P, I).

3. True or False: Suppose that $X = \{a, b, c, d\}$ and (P, I) is a rational preference with
$$P = \{(a, b), (b, c), (c, d), (a, c), (a, d), (b, d)\}$$
and
$$I = \{(a, a), (b, b), (c, c)\}.$$
Further, suppose that u and u' both represent (P, I). Then, if $u(a) - u(b) < u(c) - u(d)$, then $u'(a) - u'(b) < u'(c) - u'(d)$.

4. Explain what is wrong with the following statement: Ann prefers a to b because she assigns higher utility to a than to b.

Solutions

1. Find three utility functions that represent the rational preference relation (P, I) on $X =$

$\{a, b, c, d\}$, where

$$P = \{(a, b), (b, c), (a, c), (d, b), (d, c)\}$$

and

$$I = \{(a, a), (b, b), (c, c), (a, d), (d, a)\}.$$

1. $u(a) = u(d) = 3$, $u(b) = 2$, and $u(c) = 1$
2. $u(a) = u(d) = 300$, $u(b) = 2$, and $u(c) = 0$
3. $u(a) = u(d) = 1$, $u(b) = 0.5$, and $u(c) = 0$

2. Suppose that (P, I) is a rational preference on X and that $u : X \to \mathbb{R}$ represents (P, I).
   a. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be the function where for all $x \in \mathbb{R}$, $f(x) = 2x + 3$. Then, $f \circ u$ represents (P, I).
   True: if $u(x) \geq u(y)$ then $2u(x) + 3 \geq 2u(y) + 3$
   b. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be the function where for all $x \in \mathbb{R}$, $f(x) = -5x$. Then, $f \circ u$ represents (P, I).
   False: Suppose that $a\ P\ b$ and $u(a) = 2 > u(b) = 1$. Then $f \circ u(a) = -10 < f \circ b(b) = -5$.
   c. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be the function where for all $x \in \mathbb{R}$, $f(x) = x^2$. Then, $f \circ u$ represents (P, I).
   False: Suppose that $a\ P\ b$ and $u(a) = 2 > u(b) = -2$. Then $f \circ u(a) = 4 = f \circ b(b) = 4$.
3. True or False: Suppose that $X = \{a, b, c, d\}$ and (P, I) is a rational preference with

$$P = \{(a, b), (b, c), (c, d), (a, c), (a, d), (b, d)\}$$

and

$$I = \{(a, a), (b, b), (c, c)\}.$$

Further, suppose that u and u' both represent (P, I). Then, if $u(a) - u(b) < u(c) - u(d)$, then $u'(a) - u'(b) < u'(c) - u'(d)$.
False: Suppose that $u(a) = 4, u(b) = 3$, $u(c) = 2$, and $u(d) = 0$ and that $u'(a) = 7, u'(b) = 3$, $u'(c) = 2$, and $u'(d) = 0$. Both u and u' represent (P, I). We have that

$$u(a) - u(b) = 4 - 3 = 1 < u(c) - u(d) = 2 - 0 = 2.$$

However,

$$u'(a) - u'(b) = 7 - 4 = 3 > u'(c) = u'(d) = 2 - 0 = 2.$$

4. Explain what is wrong with the following statement: Ann prefers a to b because she assigns higher utility to a than to b.
   In standard rational choice models, a utility function u *represents* a decision maker's preference. In this case, assigning a higher utility to an object a than to b does not mean anything else except that a is preferred to b.

Chapter 11

From Ordinal to Cardinal Utility

It is not hard to see that if (P, I) is representable by a utility function (see Definition 10.1), then any utility function resulting the same ordering of the objects also represents (P, I). For instance, suppose that $X = \{a, b, c\}$ and (P, I) are rational preferences with $a\ P\ b\ P\ c$ (and $I = \{(a, a), (b, b), (c, c)\}$). Then, the following table gives three utility functions that represent (P, I)

	u_1	u_2	u_3
a	3	1000	1.0
b	2	900	0.8
c	1	-100	0.1

Indeed, any function $u : X \to \mathbb{R}$ such that $u(a) > u(b) > u(c)$ represents (P, I). This means that the *only* information about the decision maker's attitude towards the objects that these utility functions provide is the ordering of the objects. In particular, one cannot conclude the following about the decision maker's preferences about a, b and c:

1. The decision maker ranks b closer to a than to c (i.e., the difference in utility of a and b is smaller than the difference in utility of b and c).
2. The utility of b is 8-times the utility assigned to c.

Even though statement 1 is true of the utility functions u_2 and u_3, it is not true of u_1. Even though statement 3 is true of the utility function u_3, it is not true of the utility functions u_1 and u_2.

Utility functions that only represent the decision maker's ordering of the objects are called **ordinal** utility functions. In many choice situations, utility functions are intended to represent more than simply the ordering of the items. Such utility functions are called **cardinal** utility functions. There are different types of cardinal utility functions characterized by the what types of comparisons are meaningful:

1. **Interval scale**: Quantitative comparisons of objects accurately reflects differences between objects. For instance, temperature is an interval scale: the difference between 75°F and 70°F is the same as the difference between 30°F and 25°F. However, 70°F (= 21.11°C) is *not* twice as hot as 35°F (= 1.67°C).
2. **Ratio scale**: Quantitative comparisons of objects accurately reflects ratios between objects. For instance, weight is a ratio scale: 10lb ($= 4.53592kg$) is twice as much as 5lb ($= 2.26796kg$).

11.1 Linear Transformations

Utilities that are related by *linear transformations* will play an important role in this course.

Definition 11.1. Suppose that $u : X \to \mathbb{R}$ and $u' : X \to \mathbb{R}$ are two utility functions on a set X. We say that u' is a **linear transformation** of u if there are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that for all $x \in X$,

$$u'(x) = \alpha u(x) + \beta.$$

To illustrate the importance of Definition 11.1 for Rational Choice Theory, suppose that Ann's strict preference on the set $\{a, b, c\}$ is:

$$a \ P \ b \ P \ c.$$

An important question in Rational Choice Theory is how to infer Ann's preference over lotteries with prizes from the set $X = \{a, b, c\}$ given the strict preference over the set X. Clearly, she prefers the lottery $[a : 0.1, b : 0.9]$ to the lottery $[c : 1.0]$ (since a and b are both strictly preferred to c, she would prefer a chance to get either a or b to receiving c for sure). However, given only the information about Ann's strict preference over X we cannot infer how she would rank the lotteries $[a : 0.5, c : 0.5]$ and $[b : 1]$. To infer Ann's preference between the lotteries $[a : 0.5, c : 0.5]$ and $[b : 1]$, we need to know whether Ann ranks b closer to a than to c or ranks b closer to c than to a. That is, we need to know how Ann compares the *difference* between a and b with the difference between b and c. If u is a utility function representing Ann's preferences, we are interested in how she compares differences in the utilities assigned to a, b and c:

$$a\overbrace{\qquad\quad}^{u(a)-u(b)}b\underbrace{\qquad\quad}_{u(b)-u(c)}c.$$

If all we know is that u represents Ann's preference over X, then it is not meaningful to compare the utility differences $u(a) - u(b)$ with $u(b) - u(c)$. The problem is that there are different utility functions that both represent Ann's preferences, but differ in the ranking of the differences in utilities. For instance, both of the utility functions $u : X \to \mathbb{R}$ with $u(a) = 2, u(b) = 1, u(c) = 0$ and $u' : X \to \mathbb{R}$ with $u'(a) = 5, u'(b) = 4, u'(c) = 1$ represent Ann's preference on X. However, according to u, b is ranked evenly between a and c since $u(a) - u(b) = u(b) - u(c)$, but according to u', b is ranked closer to a than to c since $u(a) - u(b) < u(b) - u(c)$.

The crucial observation is that if all the utility functions that represent a decision maker's preferences are related by linear transformations, then comparisons of differences between utilities are meaningful.

Proposition 11.1. *Suppose that $u : X \to \mathbb{R}$ and $u' : X \to \mathbb{R}$ is a linear transformation of u. Then, for all $a, b, c, d \in X$,*

1. *if $u(a) - u(b) < u(c) - u(d)$, then $u'(a) - u'(b) < u'(c) - u'(d)$;*
2. *if $u(a) - u(b) > u(c) - u(d)$, then $u'(a) - u'(b) > u'(c) - u'(d)$; and*
3. *if $u(a) - u(b) = u(c) - u(d)$, then $u'(a) - u'(b) = u'(c) - u'(d)$.*

Proof. We only prove item . since the proofs of 2 and 3 are similar. Suppose that $u' : X \to \mathbb{R}$ is a linear transformation of $u : X \to \mathbb{R}$ and $a, b, c, d \in X$. Then there are real numbers $\alpha > 0$ and β such that for all $x \in X$, $u'(x) = \alpha u(x) + \beta$. Suppose that $u(a) - u(b) < u(c) - u(d)$. Then, since $\alpha > 0$, we have that $\alpha(u(a) - u(b)) > \alpha(u(c) - u(d))$. We show that $u'(a) - u'(b) < u'(c) - u'(d)$ as follows:

$$
\begin{aligned}
u'(a) - u'(b) &= (\alpha u(a) + \beta) - (\alpha u(b) + \beta) \\
&= \alpha(u(a) - u(b)) \\
&< \alpha(u(c) - u(d)) \\
&= (\alpha u(c) + \beta) - (\alpha u(d) + \beta) \\
&= u'(c) - u'(d)
\end{aligned}
$$

□

11.2 Exercises

1. Suppose that $X = \{a, b, c\}$ and that $u : X \to \mathbb{R}$ with $u(a) = 3$, $u(b) = 2$ and $u(c) = 0$. Which of the following utilities are linear transformations of u?

	a	b	c
u_1	32	22	2
u_2	0.75	0.5	0
u_3	9	4	0
u_4	-1	0	2

Part IV

Expected Utility Theory

The main readings for this section are:

- Chapter 4, Section 4.2 from Hausman, McPherson, and Satz (2020)
- Chapter 3, pp. 53 - 65 from Gaus and Thrasher (2021)
- Chapter 3, pp. 42 - 48 from Reiss (2013)

Chapter 12

Expected Utility

12.1 Expected Value

When the prizes are monetary values, a decision maker can compare lotteries using the expected value of the lottery.

Definition 12.1 (Expected value)**.** Suppose that $L = [x_1 : p_1, \dots, x_n : p_n]$ is a lottery where for $i = 1, \dots, n$, x_i is an amount of money. The **expected value** of L is defined as follows:

$$EV(L) = \sum_{i=1}^{n} p_i * x_i.$$

For example, if $L = [\$100 : 0.5, \$0 : 0.5]$, then $EV(L) = 0.5 * 100 + 0.5 * 0 = 50$. This means that you should be willing to pay up to \$50 to play this lottery. To illustrate this, you can evaluate the average payout after playing this lottery 1000 times (you can change the probabilities and payouts).

```
//| echo: false

viewof p = Inputs.form({
  prize1: Inputs.range([0, 100], {step: 1, value: 100, label: "prize 1"}),
  prize2: Inputs.range([0, 100], {step: 1, value: 0, label: "prize 2"}),
  prob: Inputs.range([0, 1], {step: 0.01, value:0.5, label: "probability of prize 1"}),
})

tex.block`
  EV([${p.prize1}:${p.prob}, ${p.prize2}: ${Math.round((1-p.prob)*100) / 100}]) = ${p.prob} * ${p.prize1
`

data_ev={
  let p1_data = {prize: "$" + p.prize1.toString(), num: 0};
  let p2_data = {prize: "$" + p.prize2.toString(), num: 0};
  let x = 0;
  for (let i = 0; i < 1000; i++) {

    let outcome = Math.random() < p.prob ? 1 : 0;
```

```
    p1_data["num"] += outcome;
    p2_data["num"] += 1-outcome;

}
  return [p1_data, p2_data];
}

md`After playing the lottery 1000 times, the average payout is ${(data_ev[0]["num"] * p.prize1 + data_ev

Plot.plot({
  marks: [
    Plot.barY(data_ev, {x: "prize", y: "num"})
  ]
})
```

12.1.1 St. Petersburg Paradox

A problem with evaluating lotteries using their expected value was noticed in the 18th century by the mathematician Nicolas Bernoulli. Bernoulli considered the following lottery: repeatedly flip a coin until it lands heads. If the coin is flipped n times, then the payout is 2^n. That is, he considered the following lottery:

$$L = [2 : \frac{1}{2}, 4 : \frac{1}{4}, 16 : \frac{1}{16}, \ldots, 2^n : \frac{1}{2^n}, \ldots]$$

> ⚠ Warning
>
> Note that the above lottery uses an *infinite* set of outcomes.

The problem is that the expected value of this lottery is infinite:

$$EV(L) = \sum_{n=1}^{\infty} \frac{1}{2^n} * 2^n = \sum_{n=1}^{\infty} 1 = \infty$$

This means that a decision make should be willing to pay *any* amount of money to play this lottery. One way to avoid this absurd conclusion is to replace the monetary value of an item with its *utility*, as explained by Daniel Bernoulli in 1738:

> the value of an item must not be based on its price, but rather on the utility it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount.

One way to make the above idea precise is to take the log base e of the prize when calculating the expected value. That is, the value of the lottery L is calculated using the *utility* rather than payout:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} * \ln(2^n) = \sum_{n=1}^{\infty} \frac{1}{2^n} * \ln(2^n) < \infty.$$

This solution is illustrated below: The graph lists the number of prizes received after playing the St. Petersburg lottery 1000 times (you can change this value). The table below the graph lists for each number of flips, the probability of observing that flip, the payout, the expected payout, the utility, and the expected payout.

See St. Petersburg Paradox, Stanford Encyclopedia of Philosophy for more discussion about variants and other solutions.

12.2 Expected Utility

Suppose that a decision maker has a lottery ticket that pays $1,000 with probability 0.5, otherwise it pays nothing. Suppose that the decision maker is offered a chance to trade this lottery ticket for $499. Is it rational for the decision maker to accept this trade? The decision maker is comparing two lotteries:

$$L_1 = [1000 : 0.5, 0 : 0.5] \quad \text{and} \quad L_2 = [499 : 1]$$

Accepting the trade means that the decision maker revealed a preference of L_2 over L_1. However, the expected value of L_1 is greater than the expected value of L_2. Thus, if the decision to trade is rational, then it must be explained using something other than the expected value of the lotteries. As discussed in the previous section, the key idea is to compare lotteries in terms of their *expected utilities* rather than their expected values.

Definition 12.2 (Expected utility)**.** Suppose that X is a set and $u : X \to \mathbb{R}$ is a utility function. If $L = [x_1 : p_1, \dots, x_n : p_n]$ is a lottery on X. Then, the expected utility of L is:

$$EU(L, u) = \sum_{i=1}^{n} p_i * u(x_i).$$

The decision to trade is rational when there is some utility function u on monetary prizes such that $EU(L_2, u) > EU(L_1, u)$. Assuming that $u(\$0) = 0$, then we have that $EU(L_2, u) > EU(L_1, u)$ for any utility function such that $u(499) > 0.5u(1000)$. For instance, if $u(x) = \sqrt(x)$, then

$$u(499) = \sqrt{499} = 22.34 > 15.81 = 0.5\sqrt{1000} = 0.5 * u(1000).$$

One motivation for the above utility function is that the decision maker is *risk-averse*. That is, the decision maker prefers a sure-thing over a risky lottery even if the sure-thing has a lower expected value. Not all utility function exhibit this type of risk-aversion. For instance, consider the three utility functions depicted in the graphs below. The blue dashed line is $u(499)$ and the red dashed line is $0.5 * u(1000)$. The utility functions on the first row are risk-averse, while the utility function on the bottom right represents a decision-maker that is not risk-averse (in fact, this utility function represents a decision maker that is *risk-seeking*).

12.3 Exercises

1. What is the expected value of $L = [\$100 : 0.2, \$60 : 0.6, \$0 : 0.1, \$10 : 0.1]$?
2. What is the expected utility of $L = [\$100 : 0.2, \$60 : 0.6, \$0 : 0.1, \$10 : 0.1]$ using the utility function where for each monetary amount m, $u(m) = \sqrt{m}$?
3. Suppose that $X = \{a, b, c\}$ and that a rational decision maker strictly prefers a to b and b to c. Assuming that the decision maker compares lotteries by comparing their expected utilities, find a utility function on X such that the decision maker strictly prefers L_2 to L_1, where

$$L_1 = [a : 0.6, c : 0.4] \text{ and } L_2 = [b : 1]$$

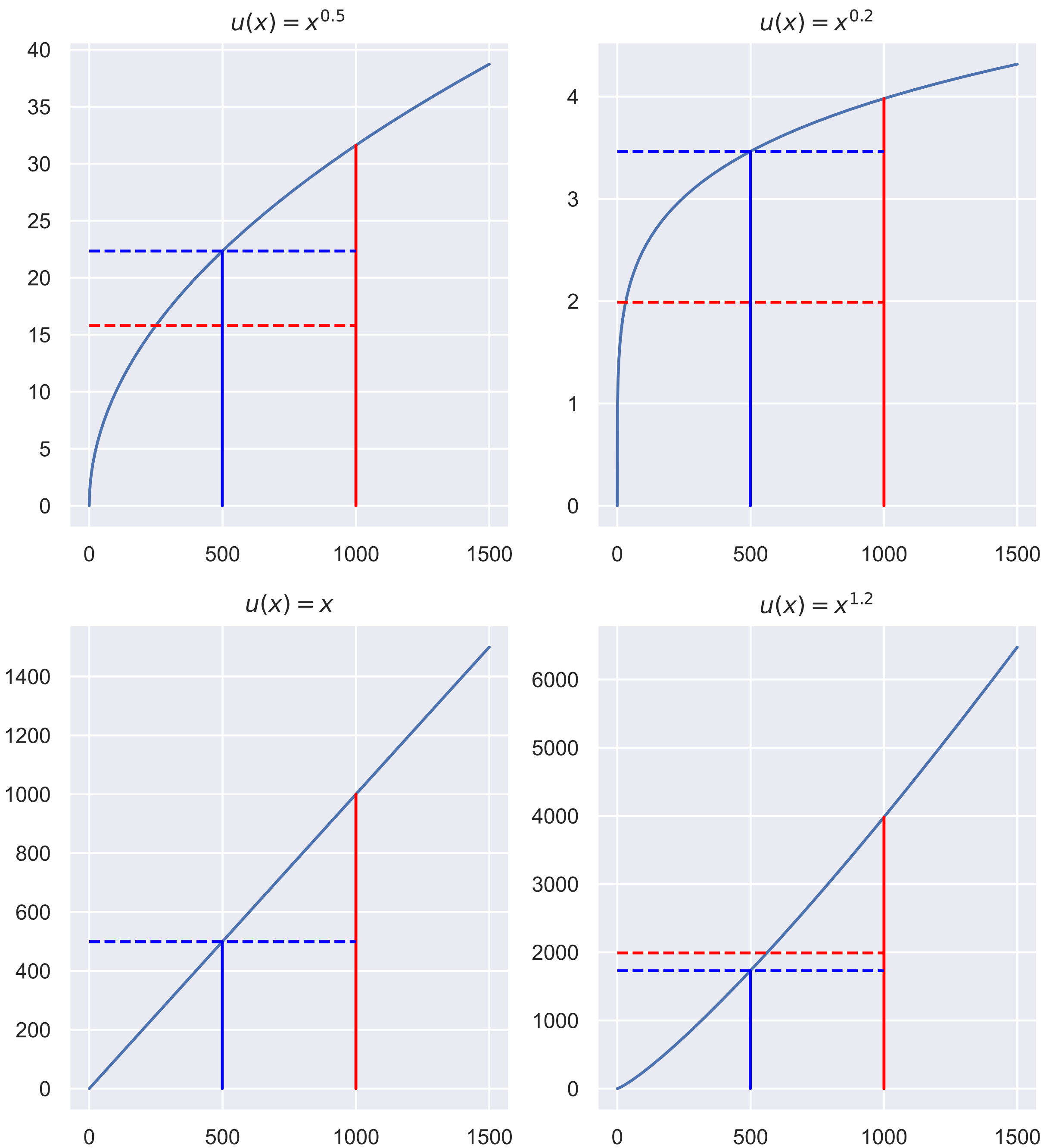


Figure 12.1: Expected utility plots.

4. Suppose that $X = \{a, b, c\}$ and that a rational decision maker strictly prefers a to b and b to c. Assuming that the decision maker compares lotteries by comparing their expected utilities, find a utility function on X such that the decision maker strictly prefers L_1 to L_2, where

$$L_1 = [a : 0.6, c : 0.4] \text{ and } L_2 = [b : 1]$$

5. Suppose that $X = \{a, b, c\}$ and that a rational decision maker strictly prefers a to b and b to c. Assuming that the decision maker compares lotteries by comparing their expected utilities, find a utility function on X such that the decision maker is indifferent between the following two lotteries:

$$L_1 = [a : 0.6, c : 0.4] \text{ and } L_2 = [b : 1]$$

6. Suppose that Ann is faced with the choice between lotteries L_1 and L_2 where:

$$L_1 = [\$4000 : 0.4,\ \$0 : 0.6] \qquad L_2 = [\$3000 : 1.0]$$

Suppose that Ann ranks L_2 over L_1 (e.g., $L_2\ P\ L_1$) and that Ann is also faced with the choice between lotteries L_3 and L_4 where:

$$L_3 = [\$4000 : 0.2, \$0 : 0.8] \qquad L_4 = [\$3000 : 0.5, \$0 : 0.5]$$

Can we conclude anything about how Ann ranks L_3 and L_4?

7. Suppose that a decision maker strictly prefers a to b and b to c. Consider the lotteries $L_1 = [a : 0.7, c : 0.3]$ and $L_2 = [c : 0.3, b : 0.7]$. Assuming that the decision maker's preferences are rational and that she compares lotteries by maximizing expected utility, which of the following is true:

   a. The decision maker strictly prefers L_1 over L_2.
   b. The decision maker strictly prefers L_2 over L_1.
   c. The decision maker is indifferent between L_1 and L_2.
   d. There is not enough information to answer this question.

Solutions

1. What is the expected value of $L = [\$100 : 0.2, \$60 : 0.6, \$0 : 0.1, \$10 : 0.1]$?
   $EV(L) = 0.2 * 100 + 0.6 * 60 + 0.1 * 0 + 0.1 * 10 = 20 + 3.6 + 0 + 1 = 24.6$
2. What is the expected utility of $L = [\$100 : 0.2, \$60 : 0.6, \$0 : 0.1, \$10 : 0.1]$ using the utility function where for each monetary amount m, $u(m) = \sqrt{m}$?
   $EU(L, u) = 0.2 * \sqrt{100} + 0.6 * \sqrt{60} + 0.1 * \sqrt{0} + 0.1 * \sqrt{10} = 6.964$
3. Suppose that $X = \{a, b, c\}$ and that a rational decision maker strictly prefers a to b and b to c. Assuming that the decision maker compares lotteries by comparing their expected utilities, find a utility function on X such that the decision maker strictly prefers L_2 to L_1, where

   $$L_1 = [a : 0.6, c : 0.4] \text{ and } L_2 = [b : 1].$$

   Let $u(a) = 1$, $u(b) = 0.7$ and $u(c) = 0$. Then

   $$EU(L_1, u) = 0.6 * 1 + 0.4 * 0 = 0.6 < EU(L_2, u) = 0.7 * 1 = 0.7$$

4. Suppose that $X = \{a, b, c\}$ and that a rational decision maker strictly prefers a to b and b to c. Assuming that the decision maker compares lotteries by comparing their expected utilities, find a utility function on X such that the decision maker strictly prefers L_1 to L_2, where

   $$L_1 = [a : 0.6, c : 0.4] \text{ and } L_2 = [b : 1].$$

Let $u(a) = 1$, $u(b) = 0.5$ and $u(c) = 0$. Then

$$EU(L_1, u) = 0.6 * 1 + 0.4 * 0 = 0.6 > EU(L_2, u) = 0.5 * 1 = 0.5$$

5. Suppose that $X = \{a, b, c\}$ and that a rational decision maker strictly prefers a to b and b to c. Assuming that the decision maker compares lotteries by comparing their expected utilities, find a utility function on X such that the decision maker is indifferent between the following two lotteries:

$$L_1 = [a : 0.6, c : 0.4] \text{ and } L_2 = [b : 1]$$

Let $u(a) = 1$, $u(b) = 0.6$ and $u(c) = 0$. Then

$$EU(L_1, u) = 0.6 * 1 + 0.4 * 0 = 0.6 = EU(L_2, u) = 0.6 * 1 = 0.6$$

6. Suppose that Ann is faced with the choice between lotteries L_1 and L_2 where:

$$L_1 = [\$4000 : 0.4,\ \$0 : 0.6] \qquad L_2 = [\$3000 : 1.0]$$

Suppose that Ann ranks L_2 over L_1 (e.g., $L_2\ P\ L_1$) and that Ann is also faced with the choice between lotteries L_3 and L_4 where:

$$L_3 = [\$4000 : 0.2, \$0 : 0.8] \qquad L_4 = [\$3000 : 0.5, \$0 : 0.5]$$

Can we conclude anything about how Ann ranks L_3 and L_4?
Ann must rank L_4 over L_3 (e.g., $L_4\ P\ L_3$).
Suppose that ranks L_2 strictly above L_1 according to expected utility. Then there is a utility $u : \{\$4000, \$0, \$3000\} \to \mathbb{R}$ where:

$$0.4 * u(\$4000) + 0.6 * u(\$0) < 1.0 * u(\$3000)$$

Then, reason as follows:

$$\begin{aligned}
&0.4 * u(\$4000) + 0.6 * u(\$0) < 1.0 * u(\$3000)\\
&0.2 * u(\$4000) + 0.3 * u(\$0) < 0.5 * u(\$3000) \quad \text{(multiply by 0.5)}\\
&0.2 * u(\$4000) + 0.8 * u(\$0) < 0.5 * u(\$3000) + 0.5u(\$0) \quad (\text{add } 0.5 * u(\$0))
\end{aligned}$$

This means that L_4 is ranked strictly above L_3 according to expected utility theory and Ann's utility $u : \{\$4000, \$0, \$3000\} \to \mathbb{R}$.

7. Suppose that a decision maker strictly prefers a to b and b to c. Consider the lotteries $L_1 = [a : 0.7, c : 0.3]$ and $L_2 = [c : 0.3, b : 0.7]$. Assuming that the decision maker's preferences are rational and that she compares lotteries by maximizing expected utility, which of the following is true:
   a. The decision maker strictly prefers L_1 over L_2.
   b. The decision maker strictly prefers L_2 over L_1.
   c. The decision maker is indifferent between L_1 and L_2.
   d. There is not enough information to answer this question.

Suppose that u is a utility function representing the decision maker's preferences.
Then $EU(L_1, u) = 0.7u(a) + 0.3u(c)$ and $EU(L_2, u) = 0.3u(c) + 0.7u(b)$
Since $u(a) > u(b)$, we have $0.7u(a) > 0.7u(b)$ and so $0.7u(a) + 0.3u(c) > 0.7u(b) + 0.3u(c)$. Hence, the decision maker must strictly prefer L_1 to L_2.

Chapter 13

Preferences over Lotteries

Suppose that X is a finite set and that $\mathcal{L}(X)$ is the set of all lotteries over X. In this section, we are interested in decision makers that have preferences over the set $\mathcal{L}(X)$. That is, the decision makers are comparing lotteries over a set X. For example, suppose that $X = \{a, b\}$ and consider three decision makers with different preferences over $\mathcal{L}(X)$:

1. Ann prefers lotteries that give a higher probability to outcome a. So, for instance, Ann has the following preferences:

$$[a:1, b:0]\ P\ [a:0.75, b:0.25]\ P\ [a:0.5, b:0.5]\ P\ [a:0.25, b:0.75]\ P\ [a:0, b:1].$$

2. Bob prefers lotteries that give a higher probability to outcome b. So, for instance, Bob has the following preferences:

$$[a:0, b:1]\ P\ [a:0.25, b:0.75]\ P\ [a:0.5, b:0.5]\ P\ [a:0.75, b:0.25]\ P\ [a:1, b:0].$$

3. Carol prefers lotteries that are closer to being a fair lottery. So, for instance, Carol has the following preferences:

$$[a:0.5, b:0.5]\ P\ [a:0.75, b:0.25]\ I\ [a:0.25, b:0.75]\ P\ [a:1, b:0]\ I\ [a:0, b:1].$$

It is not hard to see that Ann, Bob and Carol each have a rational preference over $\mathcal{L}(\{a, b\})$. As explained in Chapter 10, this means that for each decision maker there is a utility function assigning real numbers to lotteries that represents their rational preference. The following utility functions on the set of lotteries over $\{a, b\}$ represent Ann, Bob, and Carol's preferences:

1. Ann's utility function is $U_A([a:r, b:1-r]) = r$, for all $r \in [0, 1]$.
2. Bob's utility function is
   $U_B([a:r, b:1-r]) = 1 - r$, for all $r \in [0, 1]$.
3. Carol's utility function is
   $U_C([a:r, b:1-r]) = -|r - 0.5|$, for all $r \in [0, 1]$.

! Notation for utility functions

In the chapter on utility functions, we use a lowercase "u" to represent a utility function on a set X. In this section, we use a capital "U" (possibly with subscripts) to represent utility functions on lotteries. This because we need to distinguish between utility functions on the set X and utility functions on the set $\mathcal{L}(X)$.

The above utility functions are displayed in the following graph:

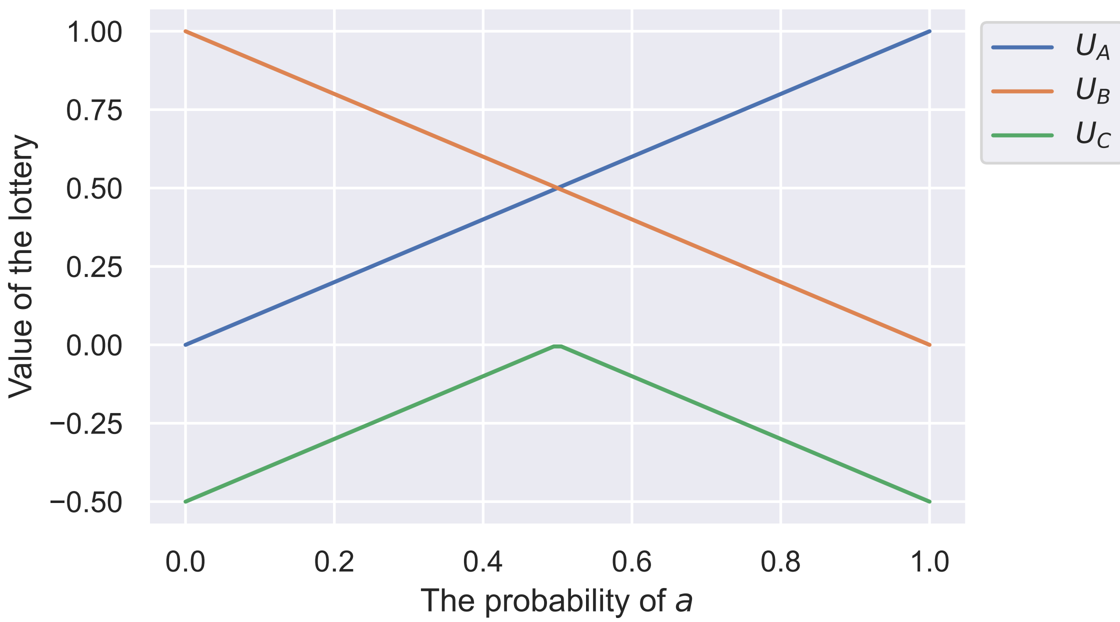


Figure 13.1: Graphs of the utility functions $U_A([a : r, b : 1 - r]) = r$, $U_B([a : r, b : 1 - r]) = 1 - r$, and $U_C([a : r, b : 1 - r]) = -|r - 0.5|$.

There is an important difference between Carol's preferences and Ann and Bob's preferences over the set of lotteries.

Both Ann and Bob's preference satisfy the following property:

Definition 13.1. Suppose that X is a finite set, $\mathcal{L}(X)$ is the set of all lotteries over X. A rational preference (P, I) over $\mathcal{L}(X)$ is **expected utility representable** provided that there is a utility function $u : X \to \mathbb{R}$ such that for all lotteries $L, L' \in \mathcal{L}(X)$,

1. if $L\ P\ L'$, then $EU(L, u) > EU(L', u)$; and
2. if $L\ I\ L'$, then $EU(L, u) = EU(L', u)$.

It is not hard to see that both Ann and Bob's preferences are expected utility representable (for Ann, consider the utility that assigns 1 to a and 0 to b, and for Bob, consider the utility function that assigns 0 to a and 1 to b). On the other hand, Carol's preference is **not** expected utility representable.

Explanation why Carol's preference is not expected utility representable.

Towards a contradiction, suppose that Carol's preferences are expected utility representable. Then, there is a utility function $u : \{a, b\} \to \mathbb{R}$ such that

1. Since $[a : 0.5, b : 0.5]\ P\ [a : 0.75, b : 0.25]$, we have that $EU([a : 0.5, b : 0.5], u) > EU([a : 0.75, b :$

$0.25], u)$. This implies that

$$\begin{aligned} 0.5 * u(a) + 0.5 * u(b) &= EU([a : 0.5, b : 0.5], u) \\ &> EU([a : 0.75, b : 0.25], u) \\ &= 0.75 * u(a) + 0.25 * u(b) \end{aligned}$$

Thus, $0.5u(a) + 0.5u(b) > 0.75u(a) + 0.25u(b)$, and so, we have that $u(b) > u(a)$.

2. Since $[a : 0.75, b : 0.25]\ I\ [a : 0.25, b : 0.75]$, we have that $EU([a : 0.75, b : 0.25], u) > EU([a : 0.25, b : 0.75], u)$. This implies that

$$\begin{aligned} 0.75 * u(a) + 0.25 * u(b) &= EU([a : 0.75, b : 0.25], u) \\ &> EU([a : 0.25, b : 0.75], u) \\ &= 0.25 * u(a) + 0.75 * u(b) \end{aligned}$$

Thus, $0.75 * u(a) + 0.25 * u(b) = 0.25 * u(a) + 0.75 * u(b)$, and so, we have that $u(a) = u(b)$.

Putting 1 and 2 together, we have that $u(b) > u(a) = u(b)$, which is impossible. Thus, Carol's preferences are not expected utility representable.

To summarize, we note the following three facts about Carol's preferences over the set of lotteries $\mathcal{L}(\{a, b\})$:

1. Carol has a rational preference on the set of lotteries $\mathcal{L}(\{a, b\})$.
2. Carol's rational preference is representable by the utility function $U_C : \mathcal{L}(\{a, b\}) \to \mathbb{R}$ where for all $r \in [0, 1]$, $U_C([a : r, b : 1 - r]) = -|r - 0.5|$.

3. Carol's rational preference is *not* expected utility representable.

The fact that Carol's rational preference is representable by U_C yet her rational preference is not expected utility representable means that U_C fails to satisfy the following important property of utility functions over lotteries.

Definition 13.2 (Linear Utility Function)**.** A utility function $U : \mathcal{L}(X) \to \mathbb{R}$ on a set of lotteries over a set X is **linear** provided that for all lotteries $L_1, \dots, L_n \in \mathcal{L}(X)$,

$$U([L_1 : p_1, \dots, L_n : p_n]) = \sum_{i=1}^{n} p_i * U(L_i).$$

For instance, the utility function U_C is not linear since $U_C([a : 0.5, b : 0.5]) \neq 0.5 * U_C([a : 1]) + 0.5 * U_C([b : 1])$:

- $U_C([a : 0.5, b : 0.5]) = -|0.5 - 0.5| = 0$
- $0.5 * U_C([a : 1]) + 0.5 * U_C([b : 1]) = 0.5 * -|1 - 0.5| + 0.5 * -|0 - 0.5| = 0.5 * -0.5 + 0.5 * -0.5 = -0.5$

The remainder of this chapter presents the additional axioms that are required in order to show that a decision maker's rational preference is represented by a *linear* utility function on the lotteries.

Chapter 14

Compound Lotteries

Consider the following two lotteries:

- $L_1 = [a : 0.5, b : 0.3, c : 0.2]$
- $L_2 = [[a : 0.4, b : 0.6] : 0.5, [a : 0.6, c : 0.4] : 0.5]$

Both lotteries L_1 and L_2 assign the same probabilities to the outcomes. That is, we can simplify lottery L_2 as follows:

$$[a : (0.5 * 0.4 + 0.5 * 0.6), b : 0.5 * 0.6, c : 0.5 * 0.4].$$

More generally, given a compound lottery L, we can define a simplified version of L:

Definition 14.1 (Simplified Lottery)**.** Suppose that $L = [L_1 : p_1, \dots, L_n, p_n]$ is a compound lottery, where for each $i = 1, \dots, n$, we have $L_i = [x_1 : p_{1,i}, \dots, x_n : p_{n,i}]$. Then the simplification of L, denoted $s(L)$, is:

$$[x_1 : (p_1 p_{1,1} + p_2 p_{1,2} + \cdots p_n p_{1,n}), \dots, x_n : (p_1 p_{n,1} + p_2 p_{n,2} + \cdots p_n p_{n,n})].$$

For example, suppose that $L = [[a : 0.2, b : 0.8] : 0.4, [b : 0.3, c : 0.7] : 0.6]$. Then,

$$s(L) = [a : 0.4 * 0.2, b : (0.4 * 0.8 + 0.6 * 0.3), c : 0.6 * 0.7] = [a : 0.08, b : 0.5, c : 0.42].$$

If a decision maker is always indifferent between a lottery L and its simplified version, then the decision maker does not get any utility from the "thrill of gambling". That is, all that matters to the decision maker when comparing lotteries is how likely she is to receive prizes that she prefers.

Compound Lottery Axiom For any lottery L, the decision maker is indifferent between L and the simplification of L. Formally, if I represents the decision maker's indifference relation, then for all lotteries L, $L\ I\ s(L)$.

14.1 Exercises

1. Find the simplifications of the following compound lotteries:

   a. $L = [[a : 0.5, b : 0.5] : 0.1, b : 0.9]$
   b. $L = [[a : 0.5, c : 0.5] : 0.1, b : 0.9]$
   c. $L = [[a : 0.5, b : 0.5] : 0.75, [a : 0.2, b : 0.8] : 0.25]$
   d. $L = [[a : 1] : 0.75, [a : 0.2, b : 0.8] : 0.25]$
   e. $L = [[a : 0.75, b : 0.25] : 0.75, [a : 0.25, b : 0.75] : 0.25]$

Solutions

1. Find the simplifications of the following compound lotteries:
   a. $L = [[a : 0.5, b : 0.5] : 0.1, b : 0.9]$

$$\begin{aligned} s(L) &= [a : (0.5 * 0.1), b : (0.5 * 0.1 + 0.9)] \\ &= [a : 0.05, b : 0.95] \end{aligned}$$

   b. $L = [[a : 0.5, c : 0.5] : 0.1, b : 0.9]$

$$\begin{aligned} s(L) &= [a : (0.5 * 0.1), c : (0.5 * 0.1), b : 0.9] \\ &= [a : 0.05, b : 0.9, c : 0.05] \end{aligned}$$

   c. $L = [[a : 0.5, b : 0.5] : 0.75, [a : 0.2, b : 0.8] : 0.25]$

$$\begin{aligned} s(L) &= [a : (0.5 * 0.75 + 0.2 * 0.25), b : (0.5 * 0.75 + 0.8 * 0.25)] \\ &= [a : 0.425, b : 0.575] \end{aligned}$$

   d. $L = [[a : 1] : 0.75, [a : 0.2, b : 0.8] : 0.25]$

$$\begin{aligned} s(L) &= [a : (1 * 0.75 + 0.2 * 0.25), b : 0.8 * 0.25] \\ &= [a : 0.8, b : 0.2] \end{aligned}$$

   e. $L = [[a : 0.75, b : 0.25] : 0.75, [a : 0.25, b : 0.75] : 0.25]$

$$\begin{aligned} s(L) &= [a : (0.75 * 0.75 + 0.25 * 0.25), b : (0.25 * 0.75 + 0.75 * 0.25)] \\ &= [a : 0.625, b : 0.375] \end{aligned}$$

Chapter 15

Independence

Suppose that $X = \{a, b, c\}$ and that a decision maker strictly prefers a to b (i.e., $a\ P\ b$). Now, consider the following two lotteries:

$$L_1 = [a : 0.6, c : 0.4] \qquad \text{and} \qquad L_2 = [b : 0.6, c : 0.4].$$

Notice that both L_1 and L_2 involve the same probability of getting item c. Indeed, the *only* difference between L_1 and L_2 is that in L_1 the outcome that will obtain with probability 0.6 is a while in L_2 the outcome that will obtain with probability 0.6 is b. It seems irrational for a decision maker to strictly prefers a to b, but not strictly prefer lottery L_1 over L_2. Imagine that a decision maker is about to play the lottery L_2 and is offered the chance to trade L_2 for L_1. There seems to be something irrational about a decision maker that strictly prefers a to b yet is unwilling to trade L_2 for L_1. The independence axiom rules out this type of irrationality. Indeed, if the decision maker strictly prefers a to b, then she should strictly prefer a 60% chance to get her preferred outcome, assuming that the other outcome is the same in both lotteries. Before stating the independence axiom formally, we note the following:

- It is important that the probabilities for item c is the same in both lotteries. Indeed, there is nothing irrational about a decision maker that strictly prefers a to b yet also strictly prefers the lottery $[b : 0.8, c : 0.2]$ to the lottery $[a : 0.05, c : 0.95]$ (this may hold for a decision maker that strictly prefers a to b, but prefers a good chance of getting outcome b to a small chance to getting outcome a).
- It is important that the only difference in the outcomes in the lotteries is a and b. Indeed, there is nothing irrational about a decision maker that strictly prefers a to b yet also strictly prefers the lottery $[b : 0.6, c : 0.4]$ to $[a : 0.6, d : 0.4]$ (this may hold for a decision maker with the strict preference $a\ P\ b\ P\ c\ P\ d$, and she prefers a lottery with a chance to get her second- and third- favorite outcomes to a lottery with a chance of getting her first and least- favorite outcome).
- We can also conclude something about the decision maker's preference of a and b given her preference about the lotteries L_1 and L_2. For instance, if the decision maker strictly prefers L_2 to L_1, then it would be irrational for the decision maker to not strictly prefer b to a.

The independence axiom generalizes the above reasoning to any compound lotteries.

Independence Axiom Suppose that $\mathcal{L}$ is a set of lotteries and (P, I) is a rational preference over $\mathcal{L}$. For all $L, L', L'' \in \mathcal{L}$ and $r \in (0, 1]$,

$$L\ P\ L' \quad \text{if, and only if,} \quad [L : r,\ L'' : (1-r)]\ P\ [L' : r,\ L'' : (1-r)].$$

$$L\ I\ L' \quad \text{if, and only if,} \quad [L : r,\ L'' : (1-r)]\ I\ [L' : r,\ L'' : (1-r)].$$

Note

To illustrate the Independence Axiom, recall the following preferences over $\mathcal{L}(\{a, b\})$ discussed in **?@sec-vnm-overview** for a decision maker that prefers lotteries that are closer to being a fair lottery:

$$[a : \frac{1}{2}, b : \frac{1}{2}]\ P\ [a : \frac{1}{4}, b : \frac{3}{4}]\ I\ [a : \frac{3}{4}, b : \frac{1}{4}]\ P\ [a : 1, b : 0]\ I\ [a : 0, b : 1]$$

Assuming the Compound Lottery Axiom, the above preference violates the Independence Axiom: Let $L = [a : \frac{1}{2}, b : \frac{1}{2}]$, $L' = [a : 1, b : 0]$, and $L'' = [a : 0, b : 1]$. Then, $L\ P\ L'$. Now, we have the following:

- $[L : \frac{1}{2}, L'' : \frac{1}{2}]\ I\ s(L) = [a : (\frac{1}{4} + 0), b : (\frac{1}{4} + \frac{1}{2})] = [a : \frac{1}{4}, b : \frac{3}{4}]$; and
- $[L' : \frac{1}{2}, L'' : \frac{1}{2}]\ I\ s(L') = [a : \frac{1}{2}, b : \frac{1}{2}]$.

Then, we have that $L\ P\ L'$, but, since $[a : \frac{1}{2}, b : \frac{1}{2}]\ P\ [a : \frac{1}{4}, b : \frac{3}{4}]$, we have that $[L' : \frac{1}{2}, L'' : \frac{1}{2}]\ P\ [L : \frac{1}{2}, L'' : \frac{1}{2}]$, contrary to the Independence Axiom.

Generalizing the above example, to show that a decision maker does *not* satisfy the Independence Axiom, there must be three lotteries L, L', and L'' and a number a number r such that $0 < r \leq 1$ such that at least one of the following is true:

1. $L\ P\ L'$, but it is not the case that $[L' : r,\ L'' : (1-r)]\ P\ [L' : r,\ L'' : (1-r)]$;
2. $[L : a,\ L'' : (1-a)]\ P\ [L' : r,\ L'' : (1-r)]$, but it is not the case that $L_1\ P\ L'$;
3. $L\ I\ L'$, but it is not the case that $[L : r,\ L'' : (1-r)]\ I\ [L' : r,\ L'' : (1-r)]$; or
4. $[L : r,\ L'' : (1-r)]\ I\ [L' : r,\ L'' : (1-r)]$, but it is not the case that $L\ I\ L'$.

15.1 Exercises

1. For all $a, b, c, d \in X$ and all $p, q, r \in (0, 1]$, if $p + q + r = 1$ and $c\ P\ d$, then $[a : p, b : q, c : r]\ P\ [a : p, b : q, d : r]$

2. Suppose that $\mathcal{L}(X)$ is a set of lotteries on a set X and that (P, I) is a rational preference on $\mathcal{L}(X)$. Using the Independence Axiom, explain why the following is true: For all $a, b \in X$ and all $r \in (0, 1)$, if $a\ P\ b$, then $[a : 1]\ P\ [a : r, b : 1 - r]$.

3. Suppose that $\mathcal{L}(X)$ is a set of lotteries on a set X and that (P, I) is a rational preference on $\mathcal{L}(X)$. Using the Independence Axiom, explain why the following is true: For all $a, b \in X$ and all $p, q \in (0, 1)$ if $p > q$ and $a\ P\ b$, then $[a : p, b : 1 - p]\ P\ [a : q, b : (1 - q)]$.

4. Suppose that $u : \{a, b, c\} \to \mathbb{R}$ is a utility function with $u(a) = 2$, $u(b) = 1$ and $u(c) = 0$. Let U be a utility function on $\mathcal{L}(X)$ where for all $L \in \mathcal{L}(X)$, $U(L) = EU(L, u) + 0.5$ if L is not a sure-thing, $U(L) = EU(L, u)$ if L is a sure-thing.

   a. Show that U is not a linear utility function on $\mathcal{L}(X)$.
   b. Show that the preference generated from this utility function violates the Independence Axiom.

5. Suppose that $u : \{a, b, c\} \to \mathbb{R}$ is a utility function with $u(a) = 2$, $u(b) = 1$ and $u(c) = 0$. Let U be a utility function on $\mathcal{L}(X)$ where for all $L \in \mathcal{L}(X)$, $U(L) = EU(L, u)$ if L is not a sure-thing, $U(L) = 2 * EU(L, u)$ if L is a sure-thing.

   a. Show that U is not a linear utility function on $\mathcal{L}(X)$.
   b. Show that the preference generated from this utility function violates the Independence Axiom.

Solutions

1. For all $a, b, c, d \in X$ and all $p, q, r \in (0, 1]$, if $p + q + r = 1$ and $c\ P\ d$, then $[a : p, b : q, c : r]\ P\ [a : p, b : q, d : r]$
   First, note that since $p + q + r = 1$, we have that $p + q = 1 - r$. Thus,
   $$\frac{p}{1-r} + \frac{q}{1-r} = \frac{p+q}{1-r} = \frac{1-r}{1-r} = 1.$$
   Thus, $[a : \frac{p}{1-r}, b : \frac{q}{1-r}]$ is a lottery.
   Then, since $c\ P\ d$ (i.e., $[c : 1]\ P\ [d : 1]$), by the Independence Axiom:
   $$[c : r, [a : \frac{p}{1-r}, b : \frac{q}{1-r}] : 1 - r]\ P\ [d : r, [a : \frac{p}{1-r}, b : \frac{q}{1-r}] : 1 - r].$$
   Note the following:
   1. $s([c : r, [a : \frac{p}{1-r}, b : \frac{q}{1-r}] : 1 - r]) = [a : p, b : q, c : r]$, and
   2. $s([d : r, [a : \frac{p}{1-r}, b : \frac{q}{1-r}] : 1 - r]) = [a : p, b : q, d : r]$.

   So, by the Compound Lottery Axiom:
   $$[a : p, b : q, c : r]\ P\ [a : p, b : q, d : r].$$
2. Suppose that $\mathcal{L}(X)$ is a set of lotteries on a set X and that (P, I) is a rational preference on $\mathcal{L}(X)$. Using the Independence Axiom, explain why the following is true: For all $a, b \in X$ and all $r \in (0, 1)$, if $a\ P\ b$, then $[a : 1]\ P\ [a : r, b : 1 - r]$.
   Since $1 = r + 1 - r$, By the Compound Lottery Axiom, we have that
   $$[a : 1]\ I\ [a : r, a : 1 - r].$$
   Then, since $[a : 1]\ P\ [b : 1]$, by the Independence Axiom, we have that:
   $$[[a : 1] : 1 - r, [a : 1] : r]\ P\ [[b : 1] : 1 - r, [a : 1] : r]$$
   .
   Since we have the following:
   1. $s([[a : 1] : 1 - r, [a : 1] : r]) = [a : r, a : 1 - r]$, and
   2. $s([[b : 1] : 1 - r, [a : 1] : r]) = [a : r, b : 1 - r]$.

   By the Compound Lottery Axiom, we have that
   $$[a : 1]\ P\ [a : r, b : 1 - r].$$
3. Suppose that $\mathcal{L}(X)$ is a set of lotteries on a set X and that (P, I) is a rational preference on $\mathcal{L}(X)$. Using the Independence Axiom, explain why the following is true: For all $a, b \in X$ and all $p, q \in (0, 1)$ if $p > q$ and $a\ P\ b$, then $[a : p, b : 1 - p]\ P\ [a : q, b : (1 - q)]$.
   Suppose that $p > q$. Then there is some $x > 0$ such that $p = q + x$.
   The first thing to note is that:
   $$\frac{q}{1-x} + \frac{1-p}{1-x} = \frac{q + (1 - (q + x))}{1-x} = \frac{1-x}{1-x} = 1.$$
   So, $[a : \frac{q}{1-x}, b : \frac{1-p}{1-x}]$ is a lottery.
   Since $a\ P\ b$ (i.e., $[a : 1]\ P\ [b : 1]$), by the Independence Axiom we have that:
   $$[[a : 1] : x, [a : \frac{q}{1-x}, b : \frac{1-p}{1-x}] : (1 - x)]\ P\ [[b : 1] : x, [a : \frac{q}{1-x}, b : \frac{1-p}{1-x}] : (1 - x)].$$
   Then, we have that:

1. $s([[a:1]:x,[a:\frac{q}{1-x},b:\frac{1-p}{1-x}]) = [a:(x+\frac{q}{1-x}*(1-x)), b:\frac{1-p}{1-x}*(1-x)] = [a:(x+q), b: 1-p] = [a:p, b:1-p]$
2. $s([[b:1]:x,[a:\frac{q}{1-x},b:\frac{1-p}{1-x}]) = [a:\frac{q}{1-x}*(1-x), b:(x+\frac{1-p}{1-x}*(1-x))] = [a:q, b: ((1-p)+x)] = [a:q, b:((1-q-x)+x)] = [a:q, b:1-q]$

So, by the Compound Lottery axiom:

$$[a:p, b:1-p] \; P \; [a:q, b:1-q].$$

4. Suppose that $u:\{a,b,c\}\to\mathbb{R}$ is a utility function with $u(a)=2$, $u(b)=1$ and $u(c)=0$. Let U be a utility function on $\mathcal{L}(X)$ where for all $L\in\mathcal{L}(X)$, $U(L)=EU(L,u)+0.5$ if L is not a sure-thing, $U(L)=EU(L,u)$ if L is a sure-thing.
   a. Show that U is not a linear utility function on $\mathcal{L}(X)$.

$$\begin{aligned} U([a:0.5,b:0.5]) &= EU([a:0.5,b:0.5],u)+0.5 \\ &= 0.5*u(a)+0.5*u(b)+0.5 \\ &= 0.5*2+0.5*1+0.5 \\ &= 2 \end{aligned}$$

$$\begin{aligned} 0.5*U([a:1])+0.5*U([b:1]) &= 0.5*EU([a:1],u)+0.5*EU([b:1],u) \\ &= 0.5*u(a)+0.5*u(b) \\ &= 0.5*2+0.5*1 \\ &= 1.5 \end{aligned}$$

   Thus, $U([a:0.5,b:0.5])\neq 0.5*U([a:1])+0.5*U([b:1])$, and so U is not a linear utility function.

   b. Show that the preference generated from this utility function violates the Independence Axiom.
   Since $U([a:0.5,b:0.5])=2=U([a:1])$, we have that

$$[a:0.5,b:0.5] \; I \; [a:1].$$

   Now, consider the compound lotteries $[[a:0.5,b:0.5]:0.4, c:0.6]$ and $[[a:1]:0.4, c:0.6]$:

$$\begin{aligned} U([[a:0.5,b:0.5]:0.4,c:0.6]) &= EU([[a:0.5,b:0.5]:0.4,c:0.6],u)+0.5 \\ &= 0.2*u(a)+0.2*u(b)+0.6*u(c)+0.5 \\ &= 0.2*2+0.2*1+0.6*0+0.5 \\ &= 1.1 \end{aligned}$$

$$\begin{aligned} U([[a:1]:0.4,c:0.6]) &= EU([[a:1]:0.4,c:0.6],u)+0.5 \\ &= 0.4*u(a)+0.6*u(c)+0.5 \\ &= 0.4*2+0.6*0+0.5 \\ &= 1.3 \end{aligned}$$

   Then,

$$[[a:1]:0.4,c:0.6] \; P \; [[a:0.5,b:0.5]:0.4,c:0.6].$$

   This contradicts the Indpendence Axiom:

1. $[a:0.5, b:0.5]\ I\ [a:1]$, but
2. it is not the case that $[[a:0.5, b:0.5]:0.4, c:0.6]\ I\ [[a:0.5, b:0.5]:0.4, c:0.6]$.

5. Suppose that $u:\{a,b,c\}\to\mathbb{R}$ is a utility function with $u(a)=2$, $u(b)=1$ and $u(c)=0$. Let U be a utility function on $\mathcal{L}(X)$ where for all $L\in\mathcal{L}(X)$, $U(L)=EU(L,u)$ if L is not a sure-thing, $U(L)=2*EU(L,u)$ if L is a sure-thing.
   a. Show that U is not a linear utility function on $\mathcal{L}(X)$.

$$\begin{aligned}U([a:0.5,b:0.5]) &= EU([a:0.5,b:0.5],u)\\ &= 0.5*u(a)+0.5*u(b)\\ &= 0.5*2+0.5*1\\ &= 1.5\end{aligned}$$

$$\begin{aligned}0.5*U([a:1])+0.5*U([b:1]) &= 0.5*2*EU([a:1],u)+0.5*2*EU([b:1],u)\\ &= 0.5*2*u(a)+0.5*2*u(b)\\ &= 0.5*2*2+0.5*2*1\\ &= 3\end{aligned}$$

   Thus, $U([a:0.5,b:0.5])\neq 0.5*U([a:1])+0.5*U([b:1])$, and so U is not a linear utility function.
   b. Show that the preference generated from this utility function violates the Independence Axiom.
   Since $U([a:0.5,c:0.5])=1<U([b:1])=2$, we have that

$$[b:1]\ P\ [a:0.5,c:0.5].$$

   Now, consider the compound lotteries $[[b:1]:0.8,c:0.2]$ and $[[a:0.5,c:0.5]:0.8,c:0.2]$:

$$\begin{aligned}U([[b:1]:0.8,c:0.2]) &= EU([[b:1]:0.8,c:0.2],u)\\ &= 0.8*u(b)+0.2*u(c)\\ &= 0.8*1+0.2*0\\ &= 0.8\end{aligned}$$

$$\begin{aligned}U([[a:0.5,c:0.5]:0.8,c:0.2]) &= EU([[a:0.5,c:0.5]:0.8,c:0.2],u)+0.5\\ &= 0.4*u(a)+0.6*u(c)\\ &= 0.4*2+0.6*0\\ &= 0.8\end{aligned}$$

   Then,

$$[[b:1]:0.8,c:0.2]\ I\ [[a:0.5,c:0.5]:0.8,c:0.2].$$

   This contradicts the Indpendence Axiom:
   1. $[b:1]\ P\ [a:0.5,c:0.5]$, but
   2. it is not the case that $[[b:1]:0.8,c:0.2]\ P\ [[a:0.5,c:0.5]:0.8,c:0.2]$.

Chapter 16

Continuity

> ⚠ Warning
>
> This section contains more advanced material and can be skipped on a first reading.

Suppose that $X = \{a, b, c\}$ and that a rational decision maker strictly prefers a to b and b to c (so item a is the favorite, item b is the second-favorite, and item c is the least-favorite). The decision maker is offered series of choices between taking b for sure, or a gamble between a and c. That is, the decision maker is asked to compare the following two lotteries for different values of $r \in [0, 1]$:

$$[b : 1] \qquad \text{vs.} \qquad [a : r, c : 1 - r].$$

When $r = 1$, the gamble is strictly preferred to the sure-thing (i.e., $[a : r, c : 1-r]\ P\ [b : 1]$) and when $r = 0$, the sure-thing is strictly preferred to the gamble (i.e., $[b : 1]\ P\ [a : r, c : 1-r]$). This means that as r ranges from 0 to 1, at some point, the preference between the sure-thing and the gamble flips. Assuming that this change of opinion is "continuous" means that there must be some value of r such that the decision maker is *indifferent* between the sure-thing $[b : 1]$ and the gamble $[a : r, c : 1 - r]$.

For example, suppose that the decision maker ranks lotteries by comparing their expected utilities for a fixed utility function $u : \{a, b, c\} \to \mathbb{R}$. In the following graph, you can choose different utilities for a, b, and c defining a utility function $u : \{a, b, c\} \to \mathbb{R}$. The graph displays the utility of a (the orange line), the utility of b (the red line), the utility of c (the green line), and $EU(L, u)$ where $L = [a : r, c : 1-r]$ as r ranges between 0 and 1 (the blue line). We also display the value of r such that $EU([a : r, c : 1 - r], u) = EU([b : 1], u)$, when it exists (the dashed red line).

```
//| echo: false

viewof a = Inputs.range([0,10], {value: 7, step:0.01,  label:  tex.block`u(a)=`}
)
viewof b = Inputs.range([0,10], {value: 4,  step:0.01, label: tex.block`u(b)=`}
)

viewof c = Inputs.range([0,10], {value: 0, step:0.01,  label: tex.block`u(c)=`}
)
```

```
p = Math.round(((b-c) / (a-c)) * 1000) / 1000;

display_p = p >= 0 && p <= 1 ? p : -1;

(c <= b && b <= a) || (a <= b && b <= c) ?
tex.block`
  EU([a:${p}, c:${Math.round((1-p)*1000) / 1000}], u) = ${Math.round((p * a + (1-p) * c) * 100) / 100} =
` : tex.block` \text{There is no } r \text{ such that } EU([a:r, c:1-r], u) = EU([b:1], u)`

data_a = [
    {"x": 0, "y": a, "type":  "utility of a"},
    {"x": 1, "y": a, "type":  "utility of a"},
]
data_b = [
    {"x": 0, "y": b, "type":  "utility of b"},
    {"x": 1, "y": b, "type":  "utility of b"},
]

data_c = [
    {"x": 0, "y": c,  "type":  "utility of c"},
    {"x": 1, "y": c, "type":  "utility of c"},
]

data_eu={
  let d_eu = [];
  let x = 0;
  for (let i = 0; i <= 10000; i++) {
    d_eu[i] = {"x": x, "y": x * a + (1-x) * c, "type":  "expected utility of L"};
    x += 0.0001;
  }
  return d_eu;
}

Plot.plot({
 style: "overflow: visible;",
  inset: 10,
  x: {
    domain:[0, 1]
  },
  y: {
    grid: true,
    domain: [0, 10]
  },
    color: {
    legend: true
  },

  marks: [
    Plot.line([{x:0, y:a, "type":  "utility of a"}, {x:1, y:a,"type":  "utility of a"}], Plot.windowY({k
```

```
    Plot.ruleX([{"x": display_p, "init_y": -0.1, "end_y": b}], {x:"x", y1: "init_y", y2: "end_y", stroke
    Plot.dot([{"x": display_p, "y": 0.3}], {x:"x", y:"y", fill: "white", r: 7}),
    Plot.text([{"x": display_p, "y": 0.3, "text": p}], {x: "x", y: "y", text: "text"}),

    Plot.line(data_b, Plot.windowY({k: 14, x: "x", y: "y", stroke: "type", strokeWidth: 2})),
    Plot.line(data_c, Plot.windowY({k: 14, x: "x", y: "y", stroke: "type",strokeWidth: 2})),
    Plot.line(data_eu, Plot.windowY({k: 30, x: "x", y: "y", stroke: "type", strokeWidth: 3})),

  ]
})
```

The Continuity Axiom ensures that a decision maker's preference are continuous in the sense described above.

Continuity Axiom Suppose that $\mathcal{L}$ is a set of lotteries and (P, I) is a rational preference over $\mathcal{L}$. For all $L, L', L'' \in \mathcal{L}$, if $L\ P\ L'\ P\ L''$, then there exists an $r \in (0, 1)$ such that

$$L'\ I\ [L : r,\ L'' : (1 - r)].$$

16.1 Exercises

1. Suppose that $u : \{a, b, c\} \to \mathbb{R}$ is a utility function with $u(a) = 2$, $u(b) = 1$ and $u(c) = 0$. Let U be a utility function on $\mathcal{L}(X)$ where for all $L \in \mathcal{L}(X)$, $U(L) = EU(L, u) + 0.5$ if L is not a sure-thing, $U(L) = EU(L, u)$ if L is a sure-thing. Show that the preference generated from this utility function violates the Continuity Axiom.
2. Suppose that $u : \{a, b, c\} \to \mathbb{R}$ is a utility function with $u(a) = 2$, $u(b) = 1$ and $u(c) = 0$. Let U be a utility function on $\mathcal{L}(X)$ where for all $L \in \mathcal{L}(X)$, $U(L) = EU(L, u)$ if L is not a sure-thing, $U(L) = 2 * EU(L, u)$ if L is a sure-thing. Show that the preference generated from this utility function violates the Continuity Axiom.

Chapter 17

The Von Neumann Morgenstern Theorem

> ⚠ Warning
>
> This section contains more advanced material and can be skipped on a first reading.

The Von Neumann Morgenstern Theorem is one of the most important results in rational choice theory. It shows that if a decision maker's preferences satisfy four axioms, then the decision maker's preferences are expected utility representable. Moreover, the utility function that represents the decision maker's preferences is unique up to linear transformations. This means that if two utility functions represent the same preferences, then they are equal up to a linear transformation.

Recall the four axioms discussed in the previous sections.

Preference (Definition 8.3) Suppose that $\mathcal{L}$ is a set of lotteries. (P, I) is a rational preference relation over $\mathcal{L}$.

Compound Lottery Axiom (Chapter 14) For any lottery L, the decision maker is indifferent between L and the simplification of L. Formally, if I represents the decision maker's indifference relation, then for all lotteries L, $L\ I\ s(L)$.

Independence Axiom (Chapter 15) Suppose that $\mathcal{L}$ is a set of lotteries and (P, I) is a rational preference over $\mathcal{L}$. For all $L, L', L'' \in \mathcal{L}$ and $r \in (0, 1]$,

$$L\ P\ L' \quad \text{if, and only if,} \quad [L : r,\ L'' : (1-r)]\ P\ [L' : r,\ L'' : (1-r)].$$

$$L\ I\ L' \quad \text{if, and only if,} \quad [L : r,\ L'' : (1-r)]\ I\ [L' : r,\ L'' : (1-r)].$$

Continuity Axiom (Chapter 16) Suppose that $\mathcal{L}$ is a set of lotteries and (P, I) is a rational preference over $\mathcal{L}$. For all $L, L', L'' \in \mathcal{L}$, if $L\ P\ L'\ P\ L''$, then there exists an $r \in (0, 1)$ such that

$$L'\ I\ [L : r,\ L'' : (1-r)].$$

Theorem 17.1 (Von Neumann Morgenstern Representation Theorem)**.** *Suppose that $\mathcal{L}$ is a set of lotteries. Then, (P, I) satisfies Preference, Compound Lotteries, Independence and Continuity if, and only if, (P, I) is represented by a linear utility function (Definition 13.2).*

Moreover, u is unique up to linear transformations: $u' : \mathcal{L} \to \mathbb{R}$ also represents (P, I) if, and only if, there are real numbers $c > 0$ and d such that for all lotteries $L \in \mathcal{L}$,

$$u'(L) = c * u(L) + d.$$

Part V

Decision Theory

The main readings for this section are:

- Section 1 and Section 3 from Briggs (2019)
- Chapter 4, Section 4.3 from Hausman, McPherson, and Satz (2020)
- Chapter 3, pp. 65 - 81 from Gaus and Thrasher (2021)
- Chapter 3, pp. 46 - 53 from Reiss (2013)

Chapter 18

Allais Paradox

Suppose that there is an urn with 100 balls. There are 89 white balls, 10 blue balls and 1 red ball. You are asked to compare two sets of lotteries:

- Question 1: Which of the following two lotteries do you prefer?
  1. Lottery 1: A ball is drawn from the urn and you win \$1 million no matter what color is drawn.
  2. Lottery 2: A ball is drawn from the urn and you win \$0 if the ball is red, \$1 million if the ball is white, and \$5 million if the ball is blue.
- Question 2: Which of the following two lotteries do you prefer?
  1. Lottery 3: A ball is drawn from the urn and you win \$1 million if the ball is red, \$0 if the ball is white, and \$1 million if the ball is blue.
  2. Lottery 4: A ball is drawn from the urn and you win 0M if the ball is red, \$0 if the ball is white, and \$5 million if the ball is blue.

> ⚠ Warning
>
> You should answer the above questions before reading further.

The Allais paradox asks decision makers to form preferences over two sets of lotteries. The first two lotteries are the following, where $1M$ means "1 million dollars" and $5M$ means "5 million dollars", $0M$ means "0 dollars":

$$L_1 = [1M : 0.01, 1M : 0.89, 1M : 0.10] \quad \text{vs.} \quad L_2 = [0M : 0.01, 1M : 0.89, 5M : 0.10].$$

Many decision makers report that they strictly prefer L_1 to L_2 (i.e., $L_1\ P\ L_2$). After reporting their preference between L_1 and L_2, decision makers are asked to compare the following two lotteries:

$$L_3 = [1M : 0.01, 0M : 0.89, 1M : 0.10] \quad \text{vs.} \quad L_4 = [0M : 0.01, 0M : 0.89, 5M : 0.10].$$

Many decision makers report that they strictly prefer L_4 to L_3 (i.e., $L_4\ P\ L_3$).

The observation of the Allais paradox is the following: While there is nothing irrational about each opinion by itself, reporting both that $L_1\ P\ L_2$ *and* $L_4\ P\ L_3$ is inconsistent with expected utility theory. That is, if a decision maker ranks lotteries by their expected utility with respect to some utility function, then:

$$L_1\ P\ L_2 \text{ if, and only if, } L_3\ P\ L_4.$$

This means that for any rational decision maker we have the following:

1. $L_1 \; P \; L_2$ and $L_3 \; P \; L_4$ is consistent with expected utility theory.
2. $L_1 \; P \; L_2$ and $L_4 \; P \; L_3$ is not consistent with expected utility theory.
3. $L_2 \; P \; L_1$ and $L_3 \; P \; L_4$ is not consistent with expected utility theory.
4. $L_2 \; P \; L_1$ and $L_4 \; P \; L_3$ is consistent with expected utility theory.

The problem is that many people report the preference $L_1 \; P \; L_2$ and $L_4 \; P \; L_3$ (these are called the *Allais preferences*), and so have preferences that are inconsistent with expected utility theory. We explain why the Allais preferences are inconsistent with expected utility theory in Section 18.1. Since, Allais preferences are inconsistent with expected utility theory, by the Von Neumann Morgenstern Theorem (Theorem 17.1), the Allais preferences must violate at least one of the axioms Compound Lottery (Chapter 14), Independence (Chapter 15), or Continuity (Chapter 16). We show that, assuming the Compound Lottery axiom and that (P, I) are a rational preference (Definition 8.3) on the set of lotteries, that the Allais preferences violate the Independence Axiom in Section 18.2.

18.1 The Allais Preferences are Inconsistent with Expect Utility Theory

Lemma 18.1. *If L_1, L_2, L_3, and L_4 are defined as in the Allais paradox, then $L_1 \; P \; L_2$ and $L_4 \; P \; L_3$ is inconsistent with expect utility theory.*

Proof. To see why $L_1 \; P \; L_2$ and $L_3 \; P \; L_4$ is inconsistent with expected utility theory, we will show that for any utility function $u : \{0M, 1M, 5M\} \to \mathbb{R}$, it is impossible that

$$EU(L_1, u) > EU(L_2, u) \quad \text{and} \quad EU(L_4, u) > EU(L_3, u).$$

Suppose that $u : \{0M, 1M, 5M\} \to \mathbb{R}$ is a utility function and that $EU(L_1, u) > EU(L_2, u)$ and $EU(L_4, u) > EU(L_3, u)$. We show that this leads to a contradiction. The expected utility calculations for L_1 and L_2 are:

$$\begin{aligned} EU(L_1, u) &= EU([1M : 0.01, 1M : 0.89, 1M : 0.10], u) \\ &= 0.01 * u(1M) + 0.89 * u(1M) + 0.10 * u(1M) \\ &= u(1M) \end{aligned}$$

$$\begin{aligned} EU(L_2, u) &= EU([0M : 0.01, 1M : 0.89, 5M : 0.10], u) \\ &= 0.01 * u(0M) + 0.89 * u(1M) + 0.10 * u(5M) \end{aligned}$$

Since $EU(L_1, u) > EU(L_2, u)$, we have that:

$$u(1M) > 0.01 * u(0M) + 0.89 * u(1M) + 0.10 * u(5M).$$

Subtracting $0.89 * u(1M)$ from both sides of the inequality gives the following:

$$0.11 * u(1M) > 0.01 * u(0M) + 0.10 * u(5M).$$

Now, the expected utility calculations for L_3 and L_4 are:

$$\begin{aligned} EU(L_3, u) &= EU([1M : 0.01, 0M : 0.89, 1M : 0.10], u) \\ &= 0.01 * u(1M) + 0.89 * u(0M) + 0.10 * u(1M) \\ &= 0.11 * u(1M) + 0.89u(0M) \end{aligned}$$

$$\begin{aligned} EU(L_4, u) &= EU([0M : 0.01, 0M : 0.89, 5M : 0.10], u) \\ &= 0.01 * u(0M) + 0.89 * u(0M) + 0.10 * u(5M) \\ &= 0.90 * u(0M) + 0.10 * u(5M) \end{aligned}$$

Since $EU(L_4, u) > EU(L_3, u)$, we have that:

$$0.90 * u(0M) + 0.10 * u(5M) > 0.11 * u(1M) + 0.89u(0M).$$

If we subtract $0.89 * u(0M)$ from both sides of the inequality, then we have that:

$$0.01 * u(0M) + 0.10 * u(5M) > 0.11 * u(1M).$$

But, this is impossible since we cannot have that:

- $0.11 * u(1M) > 0.01 * u(0M) + 0.10 * u(5M)$; and
- $0.01 * u(0M) + 0.10 * u(5M) > 0.11 * u(1M)$.

□

18.2 The Allais Preferences are Inconsistent with the Independence Axiom

Lemma 18.2. *Suppose that L_1, L_2, L_3, and L_4 are defined as in the Allais paradox, and that the decision maker satisfies the Compound Lottery axiom and that (P, I) is a rational preference (Definition 8.3) on the set of lotteries. Then, $L_1\ P\ L_2$ and $L_4\ P\ L_3$ violates the Independence Axiom.*

Proof. Suppose that L_1, L_2, L_3, and L_4 are defined as in the Allais paradox, the decision maker satisfies the Compound Lottery axiom, (P, I) is a rational preference (Definition 8.3) on the set of lotteries, and that

1. $[1M : \frac{1}{100}, 1M : \frac{89}{100}, 1M : \frac{10}{100}]\ P\ [0M : \frac{1}{100}, 1M : \frac{89}{100}, 5M : \frac{10}{100}]$, and
2. $[0M : \frac{1}{100}, 0M : \frac{89}{100}, 5M : \frac{10}{100}]\ P\ [1M : \frac{1}{100}, 0M : \frac{89}{100}, 1M : \frac{10}{100}]$.

Assume that the decision maker satisfies the Independence Axiom. We will show that this leads to a contradiction.

We first note the following two consequences of the Compound Lottery axiom:

1. Since $s([1M : \frac{1}{11}, 1M : \frac{10}{11}] : \frac{11}{100}, [1M : 1] : \frac{89}{100}) = [1M : \frac{1}{100}, 1M : \frac{89}{100}, 1M : \frac{10}{100}]$, by the Compound Lottery axiom,

$$\left[\left[1M : \frac{1}{11}, 1M : \frac{10}{11}\right] : \frac{11}{100}, [1M : 1] : \frac{89}{100}\right]\ I\ \left[1M : \frac{1}{100}, 1M : \frac{89}{100}, 1M : \frac{10}{100}\right].$$

2. Since $s([[0M : \frac{1}{11}, 5M : \frac{10}{11}] : \frac{11}{100}, [1M : 1] : \frac{89}{100}]) = [0M : \frac{1}{100}, 1M : \frac{89}{100}, 5M : \frac{10}{100}]$, by the Compound Lottery axiom,

$$\left[0M : \frac{1}{100}, 1M : \frac{89}{100}, 5M : \frac{10}{100}\right] \; I \; \left[\left[0M : \frac{1}{11}, 5M : \frac{10}{11}\right] : \frac{11}{100}, [1M : 1] : \frac{89}{100}\right]$$

By first assumption of the Allais preferences, $[1M : \frac{1}{100}, 1M : \frac{89}{100}, 1M : \frac{10}{100}] \; P \; [0M : \frac{1}{100}, 1M : \frac{89}{100}, 5M : \frac{10}{100}]$, and so by transitivity of strict preference and indifference, we have that

$$\left[\left[1M : \frac{1}{11}, 1M : \frac{10}{11}\right] : \frac{11}{100}, [1M : 1] : \frac{89}{100}\right] \; P \; \left[\left[0M : \frac{1}{11}, 5M : \frac{10}{11}\right] : \frac{11}{100}, [1M : 1] : \frac{89}{100}\right].$$

By the Independence Axiom, where L is $[1M : \frac{1}{11}, 1M : \frac{10}{11}]$, L' is $[0M : \frac{1}{11}, 5M : \frac{10}{11}]$, L'' is $[1M : 1]$, and r is $\frac{11}{100}$, this implies that

$$\left[1M : \frac{1}{11}, 1M : \frac{10}{11}\right] \; P \; \left[0M : \frac{1}{11}, 5M : \frac{10}{11}\right].$$

Applying the Independence Axiom a second time where L, L', and r are as above, and L'' is $[0M : 1]$, we have that

$$\left[\left[1M : \frac{1}{11}, 1M : \frac{10}{11}\right] : \frac{11}{100}, [0M : 1] : \frac{89}{100}\right] \; P \; \left[\left[0M : \frac{1}{11}, 5M : \frac{10}{11}\right] : \frac{11}{100}, [0M : 1] : \frac{89}{100}\right].$$

To see the contradiction, note the following two consequences of the Compound Lottery axiom:

1. Since $s([1M : \frac{1}{11}, 1M : \frac{10}{11}] : \frac{11}{100}, [0M : 1] : \frac{89}{100}]) = [1M : \frac{1}{100}, 0M : \frac{89}{100}, 1M : \frac{10}{100}]$, by the Compound Lottery axiom,

$$\left[1M : \frac{1}{100}, 0M : \frac{89}{100}, 1M : \frac{10}{100}\right] \; I \; \left[\left[1M : \frac{1}{11}, 1M : \frac{10}{11}\right] : \frac{11}{100}, [0M : 1] : \frac{89}{100}\right].$$

2. Since $s([[0M : \frac{1}{11}, 5M : \frac{10}{11}] : \frac{11}{100}, [0M : 1] : \frac{89}{100}]) = [0M : \frac{1}{100}, 0M : \frac{89}{100}, 5M : \frac{10}{100}]$, by the Compound Lottery axiom,

$$\left[\left[0M : \frac{1}{11}, 5M : \frac{10}{11}\right] : \frac{11}{100}, [0M : 1] : \frac{89}{100}\right] \; I \; \left[0M : \frac{1}{100}, 0M : \frac{89}{100}, 5M : \frac{10}{100}\right]$$

Since $[[1M : \frac{1}{11}, 1M : \frac{10}{11}] : \frac{11}{100}, [0M : 1] : \frac{89}{100}] \; P \; [[0M : \frac{1}{11}, 5M : \frac{10}{11}] : \frac{11}{100}, [0M : 1] : \frac{89}{100}]$, by transitivity of strict preference and indifference, we have that

$$\left[1M : \frac{1}{100}, 0M : \frac{89}{100}, 1M : \frac{10}{100}\right] \; P \; \left[0M : \frac{1}{100}, 0M : \frac{89}{100}, 5M : \frac{10}{100}\right].$$

But, this contradictions the second assumption about the Allais preferences that $[0M : \frac{1}{100}, 0M : \frac{89}{100}, 5M : \frac{10}{100}] \; P$ $[1M : \frac{1}{100}, 0M : \frac{89}{100}, 1M : \frac{10}{100}]$.

$\square$

Chapter 19

Ellsberg Paradox

Suppose that there is an urn with 90 balls. You are told that there are 30 blue balls in the urn and that the remaining 60 balls are either yellow or green. You are asked to compare two sets of lotteries:

- Question 1: Which of the following two lotteries do you prefer?
  1. Lottery 1: A ball is drawn from the urn and you win \$1 million if the ball is blue and \$0 if the ball is yellow or green.
  2. Lottery 2: A ball is drawn from the urn and you win \$1 million if the ball is yellow and \$0 if the ball is blue or green.
- Question 2: Which of the following two lotteries do you prefer?
  1. Lottery 3: A ball is drawn from the urn and you win \$1 million if the ball is blue or green and \$0 if the ball is yellow.
  2. Lottery 4: A ball is drawn from the urn and you win \$1 million if the ball is yellow or green and \$0 if the ball is blue.

> ⚠ Warning
>
> You should answer the above questions before reading further.

The Ellsberg paradox asks decision makers to form preferences over two sets of lotteries. The difficulty with answering these questions is that the probabilities in the lotteries are unknown. Let b be an integer such that $30 \leq b \leq 60$ representing the number of blue balls and $y = 90 - 30 - b$ the number of yellow balls, $0M$ mean "0 dollars", and $1M$ mean "1 million dollars". Then, the first question asks decision makers to compare the following two lotteries:

$$L_1 = [1M : \frac{30}{90}, 0M : \frac{b}{90}, 0M : \frac{y}{90}] \quad \text{vs.} \quad L_2 = [0M : \frac{30}{90}, 1M : \frac{b}{90}, 0M : \frac{y}{90}].$$

Many decision makers report that they strictly prefer L_1 to L_2 (i.e., $L_1 \ P \ L_2$). After reporting their preference between L_1 and L_2, decision makers are asked to compare the following two lotteries:

$$L_3 = [1M : \frac{30}{90}, 0M : \frac{b}{90}, 1M : \frac{y}{90}] \quad \text{vs.} \quad L_4 = [0M : \frac{30}{90}, 1M : \frac{b}{90}, 1M : \frac{y}{90}].$$

Many decision makers report that they strictly prefer L_4 to L_3 (i.e., $L_4 \ P \ L_3$).

The observation of the Ellsberg paradox is the following: While there is nothing irrational about each opinion by itself, reporting both that $L_1\ P\ L_2$ *and* $L_4\ P\ L_3$ is inconsistent with expected utility theory. That is, if a decision maker ranks lotteries by their expected utility with respect to some utility function, then:

$$L_1\ P\ L_2 \text{ if, and only if, } L_3\ P\ L_4.$$

This means that for any rational decision maker we have the following:

1. $L_1\ P\ L_2$ and $L_3\ P\ L_4$ is consistent with expected utility theory.
2. $L_1\ P\ L_2$ and $L_4\ P\ L_3$ is not consistent with expected utility theory.
3. $L_2\ P\ L_1$ and $L_3\ P\ L_4$ is not consistent with expected utility theory.
4. $L_2\ P\ L_1$ and $L_4\ P\ L_3$ is consistent with expected utility theory.

19.1 The Ellsberg Preferences are Inconsistent with Expect Utility Theory

Lemma 19.1. *If L_1, L_2, L_3, and L_4 are defined as in the Ellsberg paradox, then $L_1\ P\ L_2$ and $L_4\ P\ L_3$ is inconsistent with expect utility theory.*

Proof. To see why $L_1\ P\ L_2$ and $L_3\ P\ L_4$ is inconsistent with expected utility theory, we will show that for any utility function $u : \{0M, 1M, 5M\} \to \mathbb{R}$, it is impossible that

$$EU(L_1, u) > EU(L_2, u) \quad \text{and} \quad EU(L_4, u) > EU(L_3, u).$$

Suppose that $u : \{0M, 1M\} \to \mathbb{R}$ is a utility function and that $EU(L_1, u) > EU(L_2, u)$ and $EU(L_4, u) > EU(L_3, u)$. We show that this leads to a contradiction. The expected utility calculations for L_1 and L_2 are:

$$\begin{aligned}
EU(L_1, u) &= EU(\left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 0M : \frac{y}{90}\right], u) \\
&= \frac{30}{90} * u(1M) + \frac{b}{90} * u(0M) + \frac{y}{90} * u(0M) \\
&= \frac{30}{90} * u(1M) + \frac{b+y}{90} * u(0M)
\end{aligned}$$

$$\begin{aligned}
EU(L_2, u) &= EU(\left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 0M : \frac{y}{90}\right], u) \\
&= \frac{30}{90} * u(0M) + \frac{b}{90} * u(1M) + \frac{y}{90} * u(0M) \\
&= \frac{30+y}{90} * u(0M) + \frac{b}{90} * u(1M)
\end{aligned}$$

Since $EU(L_1, u) > EU(L_2, u)$, we have that:

$$\frac{30}{90} * u(1M) + \frac{b+y}{90} * u(0M) > \frac{30+y}{90} * u(0M) + \frac{b}{90} * u(1M)$$

Subtracting $\frac{30}{90} * u(1M)$ and $\frac{30+y}{90} * u(0M)$ from both sides of the inequality gives the following:

$$\frac{b + y - 30 - y}{90} * u(0M) > \frac{b - 30}{90} * u(1M)$$

Simplifying the probabilities, we have that:

$$\frac{b - 30}{90} * u(0M) > \frac{b - 30}{90} * u(1M)$$

Now, the expected utility calculations for L_3 and L_4 are:

$$\begin{aligned} EU(L_3, u) &= EU(\left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 1M : \frac{y}{90}\right], u) \\ &= \frac{30}{90} * u(1M) + \frac{b}{90} * u(0M) + \frac{y}{90} * u(1M) \\ &= \frac{30 + y}{90} * u(1M) + \frac{b}{90} * u(0M) \end{aligned}$$

$$\begin{aligned} EU(L_4, u) &= EU(\left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 1M : \frac{y}{90}\right], u) \\ &= \frac{30}{90} * u(0M) + \frac{b}{90} * u(1M) + \frac{y}{90} * u(1M) \\ &= \frac{30}{90} * u(0M) + \frac{b + y}{90} * u(1M) \end{aligned}$$

Since $EU(L_4, u) > EU(L_3, u)$, we have that:

$$\frac{30}{90} * u(0M) + \frac{b + y}{90} * u(1M) > \frac{30 + y}{90} * u(1M) + \frac{b}{90} * u(0M)$$

Subtracting $\frac{30+y}{90} * u(1M)$ and $\frac{30}{90} * u(0M)$ from both sides of the inequality gives the following:

$$\frac{b + y - 30 - y}{90} * u(1M) > \frac{b - 30}{90} * u(0M)$$

Simplifying the probabilities, we have that:

$$\frac{b - 30}{90} * u(1M) > \frac{b - 30}{90} * u(0M)$$

But, this is impossible since we cannot have that:

- $\frac{b-30}{90} * u(0M) > \frac{b-30}{90} * u(1M)$, and
- $\frac{b-30}{90} * u(1M) > \frac{b-30}{90} * u(0M)$.

□

19.2 The Ellsberg Preferences are Inconsistent with the Independence Axiom

Lemma 19.2. *Suppose that L_1, L_2, L_3, and L_4 are defined as in the Ellsberg paradox, and that the decision maker satisfies the Compound Lottery axiom and that (P, I) is a rational preference (Definition 8.3) on the set of lotteries. Then, $L_1\ P\ L_2$ and $L_4\ P\ L_3$ violates the Independence Axiom.*

Proof. Suppose that L_1, L_2, L_3, and L_4 are defined as in the Ellsberg paradox, the decision maker satisfies the Compound Lottery axiom, (P, I) is a rational preference (Definition 8.3) on the set of lotteries, and that the decision maker has the Ellsberg preferences:

1. $\left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 0M : \frac{y}{90}\right]\ P\ \left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 0M : \frac{y}{90}\right]$, and
2. $\left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 1M : \frac{y}{90}\right]\ P\ \left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 1M : \frac{y}{90}\right]$.

Assume that the decision maker satisfies the Independence Axiom. We will show that this leads to a contradiction.

We first note the following two consequences of the Compound Lottery axiom:

1. Since $s(\left[\left[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [0M : 1] : \frac{y}{90}\right]) = \left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 0M : \frac{y}{90}\right]$, by the Compound Lottery axiom,
$$\left[\left[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [0M : 1] : \frac{y}{90}\right]\ I\ \left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 0M : \frac{y}{90}\right].$$

2. Since $s(\left[\left[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [0M : 1] : \frac{y}{90}\right]) = \left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 0M : \frac{y}{90}\right]$, by the Compound Lottery axiom,
$$\left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 0M : \frac{y}{90}\right]\ I\ \left[\left[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [0M : 1] : \frac{y}{90}\right].$$

By first assumption of the Ellsberg preferences, $\left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 0M : \frac{y}{90}\right]\ P\ \left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 0M : \frac{y}{90}\right]$, and so by transitivity of strict preference and indifference, we have that

$$\left[\left[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [0M : 1] : \frac{y}{90}\right]\ P\ \left[\left[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [0M : 1] : \frac{y}{90}\right].$$

By the Independence Axiom, where L is $\left[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}\right]$, L' is $\left[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}\right]$, L'' is $[0M : 1]$, and r is $\frac{30+b}{90}$, this implies that

$$\left[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}\right]\ P\ \left[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}\right].$$

Applying the Independence Axiom a second time where L, L', and r are as above, and L'' is $[1M : 1]$, we have that

$$\left[\left[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [1M : 1] : \frac{y}{90}\right]\ P\ \left[\left[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [1M : 1] : \frac{y}{90}\right].$$

To see the contradiction, note the following two consequences of the Compound Lottery axiom:

1. Since $s(\left[\left[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [1M : 1] : \frac{y}{90}\right]) = \left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 1M : \frac{y}{90}\right]$, by the Compound Lottery axiom,
$$\left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 1M : \frac{y}{90}\right]\ I\ \left[\left[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [1M : 1] : \frac{y}{90}\right].$$

2. Since $s([[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}] : \frac{30+b}{90}, [1M : 1] : \frac{y}{90}]) = [0M : \frac{30}{90}, 1M : \frac{b}{90}, 1M : \frac{y}{90}]$, by the Compound Lottery axiom,

$$\left[\left[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}\right] : \frac{30+b}{90}, [1M : 1] : \frac{y}{90}\right] \; I \; \left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 1M : \frac{y}{90}\right].$$

Since $[[1M : \frac{30}{30+b}, 0M : \frac{b}{30+b}] : \frac{30+b}{90}, [1M : 1] : \frac{y}{90}] \; P \; [[0M : \frac{30}{30+b}, 1M : \frac{b}{30+b}] : \frac{30+b}{90}, [1M : 1] : \frac{y}{90}]$, by transitivity of strict preference and indifference, we have that

$$\left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 1M : \frac{y}{90}\right] \; P \; \left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 1M : \frac{y}{90}\right].$$

But, this contradictions the second assumption about the Ellsberg preferences that

$$\left[0M : \frac{30}{90}, 1M : \frac{b}{90}, 1M : \frac{y}{90}\right] \; P \; \left[1M : \frac{30}{90}, 0M : \frac{b}{90}, 1M : \frac{y}{90}\right].$$

□

Chapter 20

Decision Matrices

The basic building blocks of a decision problem are the following three sets:

1. the set of acts (also called the alternatives);

2. the set of outcomes (also called the consequences); and
3. the set of states.

Suppose that A is the set of acts, O is the set of outcomes, and S is the set of states in some decision problem. An act together with a state leads to an outcome. More formally, each act $a \in A$ is a function $a : S \to O$ associating states with outcomes. For $a \in A$, $s \in S$ and $o \in O$, we write $a(s) = o$ when act a and state s results in outcome o. The standard assumption in rational choice theory is that a decision maker chooses an element from the set A of acts and that this choice depends on which outcome is desired and beliefs about the states.

To illustrate the above ideas, suppose that you are offered a choice between two bets:

- bet 1: you receive \$100 if it rains tomorrow at noon, and
- bet 2: you receive \$200 if it does not rain tomorrow at noon.

This decision problem can be visualized using the following table, where the columns are labeled by the states, the rows are labeled by the acts and each cell of the table is the outcome that results from the chosen act and the realized state.

	rain at noon tomorrow	does not rain at noon tomorrow
bet 1	win \$100	receive nothing
bet 2	receive nothing	win \$200

The act you will choose (either bet 1 or bet 2) depends on your preferences over the outcomes (presumably you prefer more money to less) *and* your beliefs about whether or not it will rain tomorrow at noon.

Typically, when describing a decision problem it is straightforward to write down the set of acts and the set of outcomes. However, there are often multiple ways to describe the states in a decision problem. For example, one might split the state "rain at noon tomorrow" into two states: the first state is that it rains between 11am and 1pm and the second state is that it rains between 11:30am and 1:30pm. Similarly, the state "does not rain at noon tomorrow" may be split into two states: the first state is that it rains between 1pm and 2pm and the second state is that it does not rain at all. This way of describing the decision problem leads to the following table:

	rain 11am-1pm	rain 11:30am-1:30pm	rain 1:00pm-2:00pm	does not rain
bet 1	win \$100	win \$100	receive nothing	receive nothing
bet 2	receive nothing	receive nothing	win \$200	win \$200

In general, there is no single best way to describe the states in a decision problem. There are two important assumptions about states. The first assumption is that a state resolves all remaining uncertainty, so that a state together with an act results in a single outcome. So, for example, "it is cloudy at noon tomorrow" cannot be used as a state since it does not specify the outcome associated with each act. The second assumption is that the player's choice of act does not influence which state is realized. For instance, consider the following representation of the above decision problem:

	choose the correct bet	choose the wrong bet
bet 1	win \$100	receive nothing
bet 2	win \$200	receive nothing

This representation of the decision problem implies that bet 2 is clearly better than bet 1. The problem is that which of the two states is realized depends on whether it rains tomorrow at noon *and* the act chosen by the decision maker.

In these notes, we study two types of decision problems.

1. Decisions under certainty: The decision maker knows which state is realized (or, equivalently, there is only one state). In this case, we can simplify the description of the decision problem by assuming that the decision maker directly chooses from the set of outcomes.

2. Decisions under uncertainty: The decision maker is uncertain about the which state is realized. Thus, both the decision maker's preferences over the outcomes and the decision maker's beliefs about the states are represented in a decision under uncertainty.

Chapter 21

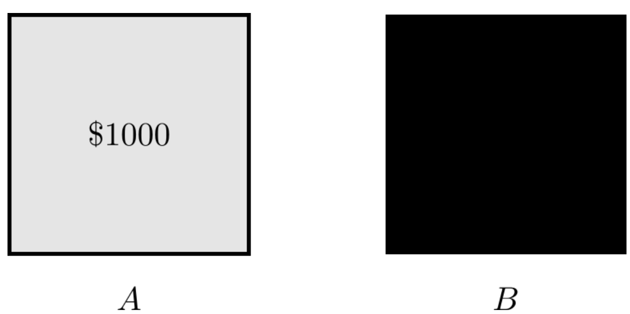
Newcomb's Paradox

R. Nozick (1969). Newcomb's Problem and Two Principles of Choice in Essays in Honor of Carl G. Hempel, Nicholas Rescher (ed.), Springer.

There are two boxes in front of us:

1. box A, which contains \$1,000;
2. box B, which contains either \$1,000,000 or nothing.

You can see inside box A, but not inside box B:



You are offered two choices:

- **one-box**: choose only box B
- **two-box**: choose both box A and box B

You can keep whatever is inside any box that is opened, but you do not get to keep what is inside a box that is not opened.

A very powerful being, called the Predictor, who has been *invariably accurate* in its predictions about your behavior in the past, has already acted in the following way:

1. If the Predictor has predicted that you will open only box B, the being has put \$1,000,000 in box B.
2. If the Predictor has predicted that you will open both boxes, the being has put nothing in box B.

Do you choose **one-box** or **two-box**?

⚠ Warning

You should answer the above question before reading further.

The decision problem described above can be formalized as follows:

- The actions are **one-box** (selecting only box B) and **two-box** (selecting both boxes);

- the outcomes are $1M$ (you receive 1 million dollars), $1T$ (you receive 1 thousand dollars), 0 (you receive nothing), and $1M + 1T$ (you receive one million one thousand dollars); and
- the states are *pred_B* meaning that the Predictor predicted that you would choose only box B, and *pred_AB* meaning that the Predictor predicted that you would choose both boxes.

Then, the decision matrix representing the above decision problem is:

	pred_B	*pred_AB*
one-box	$1M$	0
two-box	$1M$+$1T$	$1T$

There are two ways to reason about which action you should choose.

1. **two-box** should be chosen: There are two possible states *pred_B* (the prediction is that you will choose only box B) and *pred_AB* (the prediction is that you will choose both boxes). If the state is *pred_B*, then **two-box** gives the outcome $1T$ which is strictly greater than the **one-box** outcome of 0. If the state is *pred_AB*, then **two-box** gives the outcome $1M + 1T$ which is strictly greater than the **one-box** outcome of $1M$. In both cases, **two-box** gives a strictly better outcome than **one-box**, so **two-box** should be chosen.

2. **one-box** should be chosen: Let B mean that you have chosen the action **one-box** and AB mean that you have chosen the action **two-box**. To calculate the expected utilities of the actions, we consider the following (conditional) probabilities:

   - $Pr(pred\_B \mid B)$: The probability that the wizard predicted you would choose box B *given that you decided to choose box B.*

   - $Pr(pred\_AB \mid B)$: The probability that the wizard predicted you would choose both boxes *given that you decided to choose box B.*

   - $Pr(pred\_B \mid AB)$: The probability that the wizard predicted you would choose box B *given that you decided to choose both boxes.*

   - $Pr(pred\_AB \mid AB)$: The probability that the wizard predicted you would choose both boxes *given that you decided to choose both boxes.*

Then, the expected utilities (assuming, for simplicity, that the utility of $1M$ is $1,000,000$, the utility of 0 is 0, the utility of $1M + 1T$ is $1,001,000$ and the utility of $1T$ is $1,000$) of the actions are:

- $EU(\textbf{one-box}) = 1,000,000 * Pr(pred\_B \mid B) + 0 * Pr(pred\_AB \mid B)$
- $EU(\textbf{two-box}) = 1,001,000 * Pr(pred\_B \mid AB) + 1,000 * Pr(pred\_AB \mid AB)$

The assumption that the Predictor is invariably accurate about its predictions means that $Pr(pred\_B \mid B)$ and $Pr(pred\_AB \mid AB)$ are both close to 1 while $Pr(pred\_AB \mid B)$ and $Pr(pred\_B \mid AB)$ are both close to 0. This means that

$$1,000,000 * Pr(pred\_B \mid B) + 0 * Pr(pred\_AB \mid B)$$

is much greater than

$$1,001,000 * Pr(pred\_B \mid AB) + 1,000 * Pr(pred\_AB \mid AB).$$

Thus,

$$EU(\textbf{one-box}) > EU(\textbf{two-box}),$$

and so, **one-box** should be chosen.

Newcomb's paradox is interesting because it is a case in which maximizing expected utility seems to recommend an action that is *strictly dominated*. One response to Newcomb's paradox is to note that there is something odd about the expected utility calculations. In particular, the expected utility calculations assume that the prediction is *probabilistically dependent* on your choice.[1] The problem with Newcomb's paradox is that it sets up a decision problem in which the states are not independent of the actions chosen by the decision maker. A standard assumption in Rational Choice Theory is to rule out such decision problems:

Act-State Independence In any decision problem, if Pr is the probability assigned to states, X is the event that the decision maker chose action x, then for all states s, $Pr(s) = Pr(s \mid X)$. That is, the probability assigned to a state s is independent of the action chosen by the decision maker.

See Collins (1999) and Weirich (2020) for further discussion of solutions to Newcomb's paradox.

[1]There is no assumption that your choice has any *causal* influence over the prediction.

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